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THE CONTRACTION OF SATELLITE ORBITS UNDER THE INFLUENCE OF AIR DRAG

V. WITH DAY-TO-NIGHT VARIATION IN AIR DENSITY

By G. E. COOK and D. G. KING-HELE

Royal Aircraft Establishment, Farnborough

(*Communicated by M. J. Lighthill, F.R.S.—Received* 12 *February* 1965)

CONTENTS

The effect of air drag on satellite orbits of small eccentricity $(< 0{\cdot}2)$ was studied in part I on the assumption that the atmosphere was spherically symmetrical. In reality the density of the upper atmosphere depends on the elevation of the Sun above the horizon and has a maximum when the Sun is almost overhead. In the present paper the theory is extended to an atmosphere in which the air density at a given height varies sinusoidally with the geocentric angular distance from the maximum-density direction. Equations are derived which show how perigee distance and orbital period vary with eccentricity throughout the satellite's life, and how eccentricity varies with time. Expressions are also obtained for lifetime and air density at perigee in terms of the rate of change of orbital period.

The main geometrical parameter determining the long-term effect of this day-to-night variation is the angular distance ϕ_p of perigee from the maximum-density direction. Results are obtained for ϕ_p constant and ϕ_p varying linearly with time.

1. Introduction

1·1. *General*

Parts I and III of this series of papers (Cook, King-Hele & Walker 1960; King-Hele 1962) developed the basic theory for the contraction of satellite orbits in a spherically symmetrical atmosphere in which air density varied exponentially with height above the Earth (constant

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scale height). In Parts II and IV the theory was extended to take account of atmospheric oblateness (Cook, King-Hele & Walker 1961) and the variation of scale height with altitude (Cook & King-Hele 1963). In reality the density of the atmosphere above about 200 km also exhibits a day-to-night variation, and it is the aim of the present paper to incorporate this variation in the theory, in the simplest possible manner. The paper is a shortened version of a Royal Aircraft Establishment Report issued in October 1964.

1·2. *Day-to-night variation in density*

It is known that the upper-atmosphere density reaches a minimum a little before dawn, rises to a maximum at about 2 p.m. local time and then declines until about midnight, by which time the rather flat minimum is almost reached. This variation is small at a height of 200 km, less than 10 % when solar activity is near maximum, but becomes much greater at higher altitudes, with the factor

$$f = \text{maximum day-time density/minimum night-time density}$$

reaching values of up to 10 (at heights near 700 km) when the Sun is fairly active or values of up to 5 (at heights near 500 km) when solar activity is near minimum (King-Hele & Rees 1963). Although it is convenient when describing this day-to-night variation to treat it as if it depended mainly on the time of day, it really depends primarily on the zenith angle of the Sun: the atmosphere is, as it were, drawn up into a bulge having its centre beneath a 'mock Sun' which lags about 2 h or 30° behind the real Sun, and with the horizontal cross-section of the bulge being, to a first approximation, circular. Thus the upper-atmosphere density above a point on the Earth at any given latitude will experience the diurnal variation in density already described, but the centre of the bulge will only be sampled if the Sun passes nearly overhead, i.e. if the point is near the equator; at higher latitudes the maximum density attained still occurs at 2 p.m., but, since only the upper slopes of the bulge rather than its summit are being sampled, this maximum density is lower than at the same longitude nearer the equator.

Our aim is to describe analytically the evolution of satellite orbits in such an asymmetrical atmosphere, not just during one revolution, but throughout the satellite's life. This aim is difficult to achieve because of the multiplicity of parameters required to define the situation. At first sight these parameters seem likely to include (1) the factor f, (2) its variation with height, and (3) the form of the variation of density with ϕ, the angular distance from the centre of the bulge, given f; (4) the angular distance ϕ_p of perigee from the centre of the bulge, and (5), given ϕ_p, the orientation of the orbital plane relative to the bulge; the long-term motion of (6) ϕ_p and (7) the orbital plane, relative to the bulge. In fact, several of these parameters can be eliminated by various stratagems, but enough remain to complicate the presentation of some results.

1·3. *Limitations*

As before, the theory is primarily applicable to satellite orbits with perigee heights between 150 and 1000 km, and eccentricities less than 0·2: if the perigee height is below 150 km, the satellite is unlikely to remain in orbit for more than a few revolutions; at heights greater than 1000 km, solar radiation pressure exceeds air drag and is often more important as a perturbing force.

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1·4. *Previous work*

The effect of the day-to-night variation in air density on satellite orbital theory was first discussed by Wyatt (1961), who was concerned only with the modifications required in the equation for air density in terms of the rate of change of period. The problem was also investigated by Davies (1963), who assumed the contours of constant density were ellipsoids of revolution and obtained expressions for the changes in the orbital elements during one revolution, under certain restricted conditions. Fominov (1963) has obtained the changes during one revolution for a nearly spherical atmosphere in which the departures from spherical symmetry are general functions of latitude and of the geocentric sun–perigee angle.

2. Assumptions

All but two of the assumptions are virtually the same as in part I, and these are listed below without comment, since they have already been discussed in part I. Assumptions (*a*) and (*c*) are changed.

(*a*) The density ρ of the atmosphere at a given height depends on its angular distance ϕ from the centre B of the 'diurnal bulge', and B is assumed to be at the same declination as the Sun but lagging behind it in right ascension by an angle λ, normally taken as 30° (see § 3).

(*b*) The air density ρ at a given height and given ϕ does not vary with time.

(*c*) For a given value of ϕ, the air density is assumed to vary exponentially with altitude; i.e. the density scale height is taken as independent of altitude.

(*d*) The resultant aerodynamic force on the satellite acts in the direction opposite to the velocity V of the satellite relative to the ambient air, and may be taken as $\frac{1}{2}\rho V^2 SC_D$, where C_D is the drag coefficient based on cross-sectional area S, and SC_D is assumed constant.

(*e*) The atmosphere rotates with constant angular velocity w, assumed to be of the same order as that of the Earth.

(*f*) The orbit of the satellite when unperturbed by air drag is taken as an ellipse with a rotating major axis, lying in a plane through the Earth's centre, and having the Earth's centre as focus. The plane may be rotating.

(*g*) The action of air drag changes the orbit during one revolution by only a small amount whose square can be neglected.

(*h*) The orbital eccentricity is small ($< 0{\cdot}2$).

(*i*) Luni-solar perturbations are ignored.

(*j*) The inclination i of the orbit to the equator remains constant.

3. Model for air density

3·1. *General criteria governing the choice*

As stated in § 2, we assume that the air density experienced by a satellite S at a given distance r from the Earth's centre C depends only on the angular distance ϕ ($= B\hat{C}S$) from B, the 'centre of the diurnal bulge', i.e. the point where the density is greatest. It is therefore assumed that the cross-section of the bulge, in a plane perpendicular to CB, is circular; the density 30° north of B, for example, is the same as the density 30° east of B. The assumption

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of a circular cross-section is probably satisfactory as a first approximation; the exact shape is not yet known, but it is believed to be almost circular (Jacchia 1964).

In deciding what mathematical form to assume for the variation of ρ with ϕ, i.e. the density profile across the bulge, we have to remember that:

(1) the observational results so far available mostly refer to traverses of the bulge in directions fairly close to an east–west line, and, unless the cross-section is exactly circular, traverses in the north–south direction would give different profiles;

(2) the form of the profiles varies with altitude and with solar activity;

(3) the maximum amplitude of the ratio (max. density)/(min. density) varies greatly with altitude and solar activity.

In these circumstances it would be fruitless to try to specify a variation of ρ with ϕ which corresponds exactly to the particular observational profiles available, and it seems best to begin by assuming the simplest possible variation of ρ with ϕ which is reasonably realistic. A sinusoidal variation fulfils these requirements (see § 3·2 below), and gives a far better approximation to the truth than the previous assumption that density is independent of ϕ. The theory is therefore developed on the assumption that ρ varies sinusoidally with ϕ, the maximum being at $\phi = 0$ and the minimum at $\phi = 180°$. The theory will be developed in a later paper with a more realistic variation of density with ϕ and height, to check the accuracy of the simple model and to show the best way of applying it.

3·2. *Detailed specification*

The upper-atmosphere model of Harris & Priester (1962) is the most comprehensive for which values have been tabulated, and the values agree well with observation, both at conditions of high solar activity and, with some slight modifications, near sunspot minimum. The necessary modifications will be made in the CIRA 1965 model (COSPAR 1965).

Figure 1 shows the variation of density with local time as given by Harris & Priester (1962) for a height of 400 km when the solar radiation energy S' on a wavelength of 10·7 cm is 150 W m^{-2}(c/s)$^{-1}$. Since the model of Harris & Priester applies for equatorial regions, only a shift of origin is needed to convert the local time to ϕ, and a scale for ϕ has been inserted in figure 1, its zero being taken at the time of maximum density.

Also shown in figure 1 are two possible approximations for ρ, a sinusoidal variation with ϕ, that is $\rho = A + B\cos\phi\,(\equiv A - B + 2B\cos^2\tfrac{1}{2}\phi)$, and a fourth-power variation, $\rho = C + D\cos^4\tfrac{1}{2}\phi$. The values of A to D in figure 1 have been chosen to give good agreement at $\phi = 0$ and $\phi = 180°$. But it should be emphasized that for a particular orbit the values would be chosen so that the agreement was good over the section of the orbit where drag is important. Usually this section would cover quite a small angle near perigee, perhaps 60° (though for a near-circular orbit drag would be important over a wider range of values of ϕ).

From figure 1 (and similar figures, which have been drawn but are not reproduced here) it appears that the sinusoidal variation of ρ with ϕ is acceptable as a first approximation, and is far more realistic than assuming that ρ is independent of ϕ. When the approximation only needs to be accurate over a small range of values of ϕ, the sinusoidal form often proves excellent, as for example with the dotted curve, $3{\cdot}3 + 3{\cdot}3\cos\phi$, in figure 1, which is accurate

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to 2 % over a range of 180° in ϕ. The fourth-power variation gives a better fit than the sinusoidal curve and would be suitable as a second approximation, to improve the accuracy of the results obtained with the sinusoidal approximation.

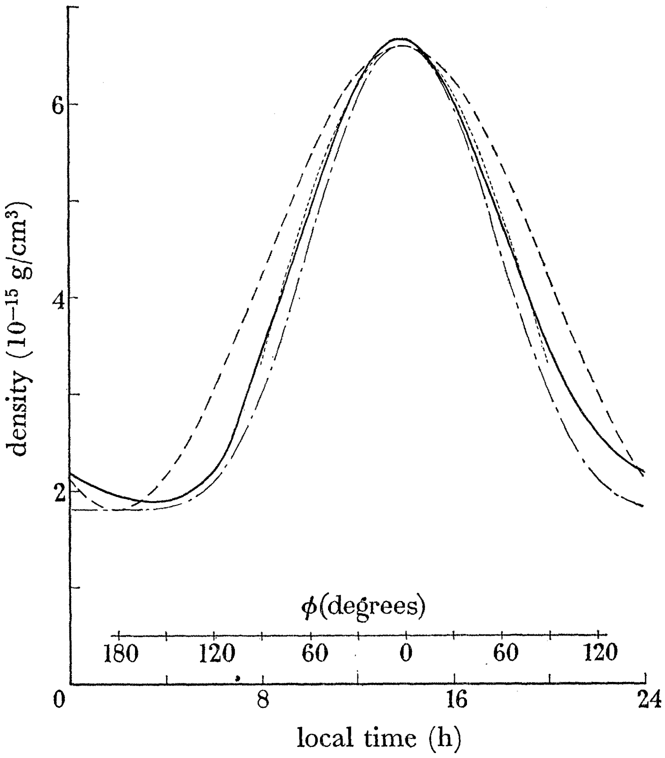

FIGURE 1. Variation of air density with time at a height of 400 km when the solar radiation energy S' on a wavelength of 10·7 cm is 150 W m^{-2}(c/s)$^{-1}$. ——, Harris & Priester (1962); – · – · –, $4{\cdot}2+2{\cdot}4\cos\phi$; ......, $3{\cdot}3+3{\cdot}3\cos\phi$; — — —, $1{\cdot}8+4{\cdot}8\cos^4\frac{1}{2}\phi$.

3·3. *Expression for air density in terms of* ϕ

If density ρ varies sinusoidally with ϕ and exponentially with distance r from the Earth's centre, we may write

$$\rho = \rho_0(1+F\cos\phi)\exp\{-(r-r_0)/H\}, \tag{1}$$

where F and H may as a first approximation be taken as constant for any given orbit, though the value of each will be chosen appropriate to a height near that of perigee and for the current level of solar activity. ρ_0 is the density at distance r_0 from the Earth's centre when $\phi = 90°$. r_0 is arbitrary, but in practice it is often most convenient to take it as either the initial or current perigee distance.

Equation (1), with H and F constant, is not intended to provide a physically consistent picture of the atmosphere: with its two adjustable constants, the equation should, however, be capable of providing a good approximation to the drag experienced by a satellite over the section of the orbit where drag is important. As an example, figure 2 shows the air density, as given by Harris & Priester (1962), along the orbit of a satellite with perigee at the centre of the bulge ($\phi = 0$) at a height of 400 km, with orbital eccentricity 0·1 and at a level of solar activity corresponding to $S' = 150$. Also shown is the approximation to density given by (1), with $\rho_0 = 4{\cdot}486\times10^{-15}$ g/cm^3, $F = 0{\cdot}48$ and $H = 67$ km; this approximation

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is never in error by more than $1\frac{1}{2}$ % of the density at perigee, and the error could no doubt be reduced by seeking better values of F and H.

The choice of the best values for F and H is beyond the scope of the present paper, since it depends on the form of their variation with height, which is not considered here and will be discussed in a future paper. However, in the absence of more sophisticated methods, a guide to the appropriate values of H and F can be obtained by assuming that values for heights near perigee should be used. For average solar activity the value of H increases

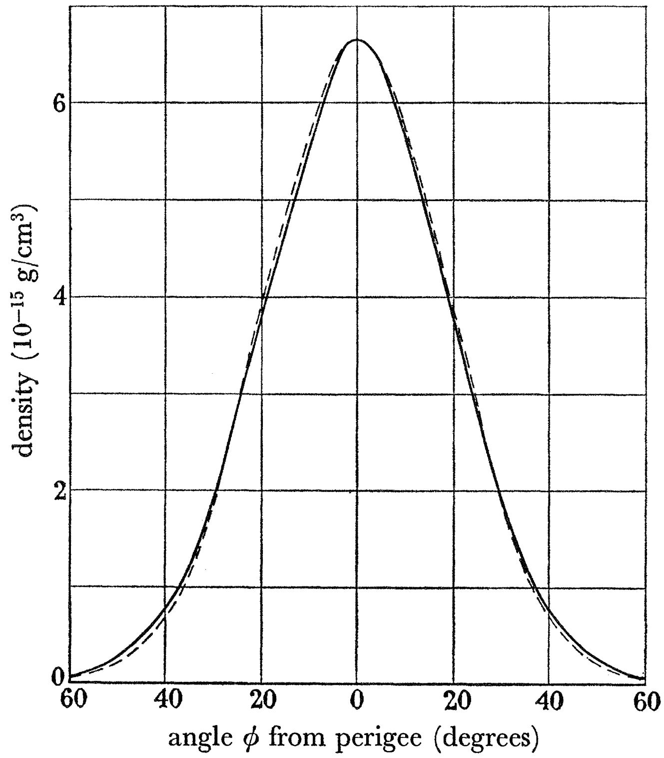

FIGURE 2. Air density at angular distances up to 60° from perigee as encountered by a satellite with perigee height 400 km, e = 0·1, S' = 150 W m^{-2}(c/s)$^{-1}$, perigee at centre of bulge. ——, Density given by Harris & Priester (1962); ------, $\rho = 4\cdot486 \times 10^{-15}\{1+0\cdot48\cos\phi\}\exp\{-(y-400)/67\}$.

from 20 km at a height of about 170 km, to 80 km at a height of about 700 km (King-Hele 1964, p. 138). The value chosen for F will depend on the angular distance of perigee from the centre of the bulge B; but possibly the best guide to F would be obtained by assuming that F was chosen so as to achieve the correct ratio of maximum density $\rho_{max.}$ to minimum density $\rho_{min.}$, i.e. so that

$$f = \frac{\rho_{max.}}{\rho_{min.}} = \frac{1+F}{1-F}, \quad \text{or} \quad F = \frac{f-1}{f+1}.$$

The variation of $(f-1)/(f+1)$ with height as given by Harris & Priester (1962) is shown in figure 3. It should be emphasized, however, that, although this value for F is perhaps the most logical one to give as a guide, the best value of F may differ widely from $(f-1)/(f+1)$.

3·4. *Relation between ϕ and true anomaly θ*

In order to evaluate the effect of air drag we need to express the angular distance ϕ of the satellite from the centre of the bulge in terms of the angular position in its orbit, as given by the true anomaly θ.

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Figure 4 (a) shows the position of the satellite S in terms of the usual orbital elements Ω, ω and i and the true anomaly θ. Let (α, δ) be its right ascension and declination. Similarly, figure 4 (b) shows the centre B of the bulge, lagging by an angle λ in right ascension behind the subsolar point, which has right ascension and declination (α_S, δ_S). Thus the right ascension and declination of the centre of the bulge (α_B, δ_B) are given by

$$\alpha_B = \alpha_S + \lambda, \tag{2}$$

$$\delta_B = \delta_S. \tag{3}$$

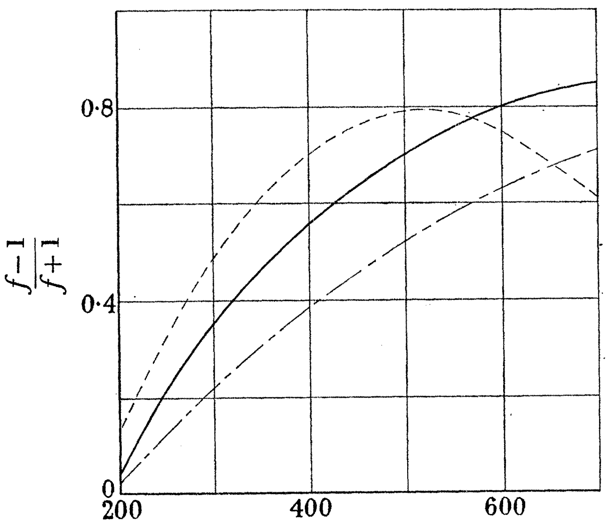

FIGURE 3. Variation of $(f-1)/(f+1)$ with height, as given by Harris & Priester (1962). ----, $S' = 70$; ——, $S' = 150$; — — —, $S' = 250$.

If we introduce a right-handed system of axes, with the x axis towards ♈, and the z axis along the Earth's axis, a unit vector **s** along OS has the components

$$\mathbf{s} = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta). \tag{4}$$

Similarly a unit vector **b** along OB has the components

$$\mathbf{b} = (\cos\delta_B\cos\alpha_B, \cos\delta_B\sin\alpha_B, \sin\delta_B). \tag{5}$$

The geocentric satellite–bulge angle $\phi = S\hat{O}B$ is given by

$$\cos\phi = \mathbf{b}.\mathbf{s} = \sin\delta\sin\delta_B + \cos\delta\cos\delta_B\cos(\alpha-\alpha_B). \tag{6}$$

Applying the equations of spherical trigonometry to the spherical triangle NSA in figure 4 (a), we obtain

$$\sin\delta = \sin i\sin(\omega+\theta), \tag{7}$$

$$\cos(\omega+\theta) = \cos(\alpha-\Omega)\cos\delta, \tag{8}$$

$$\cos i\sin(\omega+\theta) = \sin(\alpha-\Omega)\cos\delta. \tag{9}$$

Using equations (7) to (9) in (6), we have

$$\cos\phi = \sin\delta_B\sin i\sin(\omega+\theta) + \cos\delta_B\{\cos(\Omega-\alpha_B)\cos(\omega+\theta) - \cos i\sin(\Omega-\alpha_B)\sin(\omega+\theta)\}. \tag{10}$$

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Equation (10) may be written as

$$\cos\phi = A\cos\theta + B\sin\theta, \tag{11}$$

where
$$A = \sin\delta_B \sin i \sin\omega + \cos\delta_B\{\cos(\Omega - \alpha_B)\cos\omega - \cos i \sin(\Omega - \alpha_B)\sin\omega\}, \tag{12}$$

$$B = \sin\delta_B \sin i \cos\omega - \cos\delta_B\{\cos(\Omega - \alpha_B)\sin\omega + \cos i \sin(\Omega - \alpha_B)\cos\omega\}. \tag{13}$$

Also, if ϕ_p denotes the angle $P\hat{O}B$ (the 'perigee–bulge' angle), $\phi = \phi_p$ when $\theta = 0$, so that from (11)

$$\cos\phi_p = A. \tag{14}$$

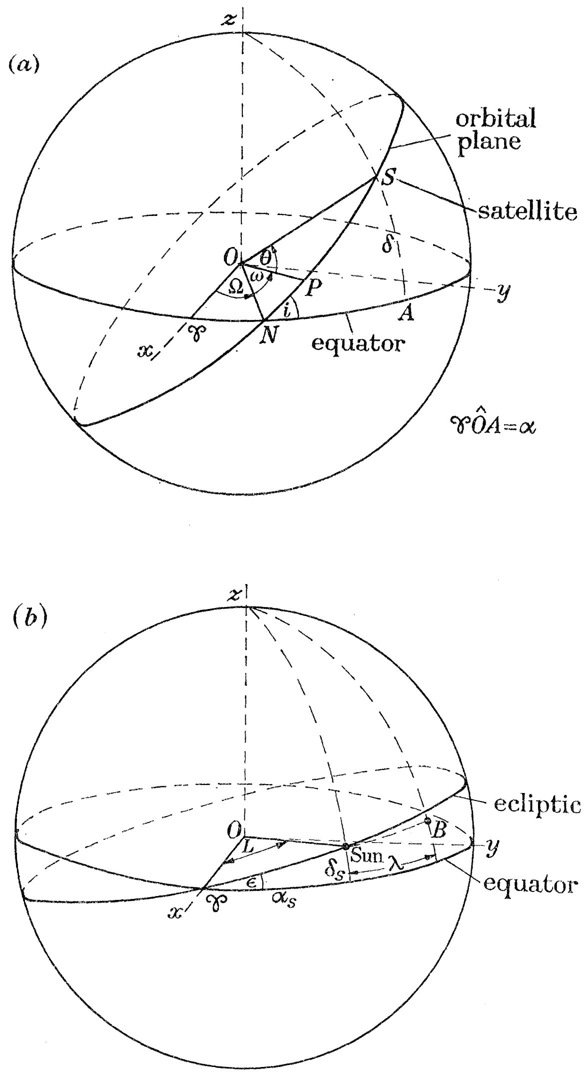

FIGURE 4. Position of (*a*) satellite, and (*b*) Sun and centre of diurnal bulge, *B*, on unit sphere.

The quantity B is not of interest, since it vanishes from the analysis, but we require to express $\cos\phi_p$ in terms of the Sun's mean longitude L. For the Sun the equations corresponding to (7)–(9) are

$$\sin\delta_S = \sin\epsilon \sin L, \tag{15}$$

$$\cos L = \cos\delta_S \cos\alpha_S, \tag{16}$$

$$\cos\epsilon \sin L = \cos\delta_S \sin\alpha_S, \tag{17}$$

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where ϵ is the obliquity of the ecliptic (23·4°). Writing $\alpha_B = \alpha_S + \lambda$ in (12), and eliminating α_S and δ_S with the aid of (15) to (17), we find

$$A = \cos\phi_p = \sin\epsilon\sin i\sin L\sin\omega + \{\cos L\cos(\Omega-\lambda) + \cos\epsilon\sin L\sin(\Omega-\lambda)\}\cos\omega$$
$$-\cos i\{\cos L\sin(\Omega-\lambda) - \cos\epsilon\sin L\cos(\Omega-\lambda)\}\sin\omega. \quad (18)$$

Equation (18) can be re-arranged in a form which shows its periodicities more clearly, as

$$\cos\phi_p = \{\cos^2\tfrac{1}{2}\epsilon\cos(\omega+\Omega-\lambda-L) + \sin^2\tfrac{1}{2}\epsilon\cos(\omega+\Omega-\lambda+L)\}\cos^2\tfrac{1}{2}i$$
$$+\{\cos^2\tfrac{1}{2}\epsilon\cos(\omega-\Omega+\lambda+L) + \sin^2\tfrac{1}{2}\epsilon\cos(\omega-\Omega+\lambda-L)\}\sin^2\tfrac{1}{2}i$$
$$+\tfrac{1}{2}\{\cos(\omega-L) - \cos(\omega+L)\}\sin i\sin\epsilon. \quad (19)$$

3·5. *Density in terms of θ or E*

The density ρ may be expressed in terms of θ by (1) and (11) as

$$\rho = \rho_0\{1 + F(A\cos\theta + B\sin\theta)\}\exp\{-\beta(r-r_0)\}, \quad (20)$$

where $\beta = 1/H$. We require to express (20) in terms of the eccentric anomaly E, and this may be done using the equations

$$r = a(1-e\cos E), \quad (21)$$

$$\cos\theta = \frac{\cos E - e}{1-e\cos E} \quad (22)$$

$$= \cos E - e(1-\cos^2 E) - e^2(1-\cos^2 E)\cos E + O(e^3) \quad (23)$$

and

$$\sin\theta = \frac{(1-e^2)^{\frac{1}{2}}\sin E}{1-e\cos E}. \quad (24)$$

Using (14), (21), (23) and (24), equation (20) becomes

$$\rho = \rho_0\left[1 + F\cos\phi_p\{\cos E - e(1-\cos^2 E) - e^2(1-\cos^2 E)\cos E + O(e^3)\}\right.$$
$$\left. + \frac{FB(1-e^2)^{\frac{1}{2}}\sin E}{1-e\cos E}\right]\exp\{\beta(r_0 - a + x\cos E)\}, \quad (25)$$

where $x = ae$.

4. Basic equations

We use as variables the semi major axis a and the quantity $x = ae$. It was shown in part I that the changes per revolution in a and x are given by

$$\Delta a = -\delta a^2 \int_0^{2\pi} \frac{(1+e\cos E)^{\frac{3}{2}}}{(1-e\cos E)^{\frac{1}{2}}}\rho\,\mathrm{d}E, \quad (26)$$

$$\Delta x = -\delta a^2 \int_0^{2\pi} (\cos E + e)\left(\frac{1+e\cos E}{1-e\cos E}\right)^{\frac{1}{2}}\rho\,\mathrm{d}E, \quad (27)$$

where

$$\delta = \frac{SC_D}{m}\left(1 - \frac{r_{p0}w}{v_{p0}}\cos i\right)^2,$$

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m being the mass of the satellite, w the angular velocity of the atmosphere, v the velocity of the satellite, and suffix $p0$ denotes initial values at perigee. Expanding (26) as a power series in e and substituting for ρ from (25), we obtain

$$\Delta a = -\delta a^2 \rho_0 \exp\{\beta(r_0 - a)\} \int_0^{2\pi} \{1 + 2e\cos E + \tfrac{3}{2}e^2\cos^2 E + O(e^3)\}$$
$$\times [1 + F\cos\phi_p\{\cos E - e(1-\cos^2 E) - e^2(1-\cos^2 E)\cos E\}]\exp(\beta x\cos E)\,\mathrm{d}E. \quad (28)$$

Since the term in B in (25) is an odd function of E, it makes no contribution to (28) and vanishes from the analysis. Similarly, from equation (27), we obtain

$$\Delta x = -\delta a^2 \rho_0 \exp\{\beta(r_0 - a)\} \int_0^{2\pi} \{\cos E + e(1+\cos^2 E) + \tfrac{1}{2}e^2(2\cos E + \cos^3 E) + O(e^3)\}$$
$$\times [1 + F\cos\phi_p\{\cos E - e(1-\cos^2 E) - e^2(1-\cos^2 E)\cos E\}]\exp(\beta x\cos E)\,\mathrm{d}E. \quad (29)$$

On multiplying out the brackets in (28) and using the integral representation of the Bessel function of the first kind and imaginary argument (Watson 1958, p. 181),

$$I_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \exp(z\cos\theta)\,\mathrm{d}\theta, \quad (30)$$

we have

$$\Delta a = -2\pi\delta a^2\rho_0 \exp\{\beta(r_0-a)\}[I_0 + 2eI_1 + \tfrac{3}{4}e^2(I_0+I_2)$$
$$+ F\cos\phi_p\{I_1 + \tfrac{1}{2}e(I_0+3I_2) + \tfrac{3}{8}e^2(I_1+3I_3)\} + O(e^3)], \quad (31)$$

where $z = \beta x = ae/H$, and I_n is written for $I_n(z)$. Similarly, from (29), we have

$$\Delta x = -2\pi\delta a^2\rho_0 \exp\{\beta(r_0-a)\}[I_1 + \tfrac{1}{2}e(3I_0+I_2) + \tfrac{1}{8}e^2(11I_1+I_3)$$
$$+ F\cos\phi_p\{\tfrac{1}{2}(I_0+I_2) + \tfrac{1}{2}e(3I_1+I_3) - \tfrac{1}{16}e^2(I_0-20I_2-5I_4)\} + O(e^3)]. \quad (32)$$

These equations demand different treatment according as $z > 3$ (phase 1) or $z < 3$ (phase 2). Section 5 deals with phase 1 and § 6 with phase 2.

5. Solution when $z > 3$ (phase 1)

5·1. *Equation for* $\mathrm{d}a/\mathrm{d}x$

When $z > 3$, i.e. when the eccentricity is greater than about 0·02, the asymptotic expansion of $I_n(z)$

$$I_n(z) \sim \frac{\exp z}{(2\pi z)^{\frac{1}{2}}}\left\{1 - \frac{4n^2-1^2}{1!\,8z} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8z)^2} - \dots\right\} \quad (33)$$

can be used in equations (31) and (32). On making this substitution and writing

$$\mu = \frac{F\cos\phi_p}{1+F\cos\phi_p} \quad (34)$$

we obtain

$$\Delta a = -\left(\frac{2\pi}{z}\right)^{\frac{1}{2}}\delta a^2\rho_p\left[1 + 2e + \tfrac{3}{2}e^2 + \frac{1}{8z} + \frac{9}{128z^2} + \frac{75}{1024z^3} - \frac{3}{4\beta a}\right.$$
$$\left. - \mu\left(\frac{1}{2z} + \frac{3}{16z^2} + \frac{45}{256z^3} + \frac{2}{\beta a}\right) + O\left(e^3, \frac{1}{z^4}, \frac{e}{z^2}\right)\right] \quad (35)$$

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and
$$\Delta x = -\left(\frac{2\pi}{z}\right)^{\frac{1}{2}}\delta a^2\rho_p\left[1+2e+\tfrac{3}{2}e^2-\frac{3}{8z}-\frac{15}{128z^2}-\frac{105}{1024z^3}-\frac{3}{4\beta a}\right.$$
$$\left.-\mu\left(\frac{1}{2z}-\frac{9}{16z^2}-\frac{75}{256z^3}+\frac{2}{\beta a}\right)+O\left(e^3,\frac{1}{z^4},\frac{e}{z^2}\right)\right], \tag{36}$$

where
$$\rho_p = \rho_0(1+F\cos\phi_p)\exp\{\beta(r_0-a)+z\}. \tag{37}$$

From (1), ρ_p represents the density at perigee, where $\phi = \phi_p$ and $r = a-Hz$. On dividing (35) by (36), writing $\beta = 1/H$, and assuming that an expansion in powers of μ is permissible, we obtain

$$\frac{da}{dx} = \frac{1}{H}\frac{da}{dz} = 1+\frac{1}{2z}+\frac{3}{8z^2}+\frac{3}{8z^3}-\frac{H}{a}-\frac{\mu}{2z^2}\left(1+\frac{3}{2z}\right)-\frac{\mu^2}{4z^3}+O\left(\frac{e}{z^2},\frac{e^2}{z},\frac{1}{z^4},\frac{\mu}{z^4},\frac{\mu^2}{z^4},\frac{\mu^3}{z^4}\right), \tag{38}$$

the terms of order $e^3, e^4, \ldots$ which ought to appear in the order term being omitted because they have zero coefficients (King-Hele 1962). Since $z > 3$, the expansion of (38) as a power series in μ is certainly valid for values of $|\mu|$ up to 1, and for our purposes it is obviously convenient to take 1 as the upper limit for $|\mu|$, since the last three order-terms can then be dropped. From (34), $|\mu|$ can only exceed 1 when $F\cos\phi_p < -\frac{1}{2}$, i.e. when F is large and $\cos\phi_p$ is negative. If $F < 0{\cdot}5$, ϕ_p is unrestricted; as F increases from 0·5 to 1, the maximum permissible value for ϕ_p decreases from 180° to 120°; as F tends to infinity, the maximum permissible ϕ_p tends to 90°. (In practice F is unlikely to exceed 1, since this would lead to negative values of density, by equation (1).)

5·2. *Solution when ϕ_p is constant*

We consider first an orbit for which the geocentric angle between perigee and the centre of the bulge, ϕ_p, remains constant, or varies so little that a constant mean value is acceptable. Since perigee height will not change by more than about one scale height during phase 1, a constant value of F is also likely to be acceptable. Hence μ will be taken constant, with $|\mu| < 1$. In practice ϕ_p remains nearly constant when the orbital inclination is near 40° or when the lifetime is short: as a very rough approximation, for $i < 50°$ and a typical orbit, ϕ_p will vary by 30° in a time of roughly $|1/(40-i)|$ years (see King-Hele & Walker 1961).

5·2·1. *Perigee distance and orbital period in terms of e*

With μ constant, equation (38) can be integrated to give

$$\frac{a_0-a}{H} = z_0-z+\tfrac{1}{2}\ln\frac{z_0}{z}+\frac{3}{8}\left(\frac{1}{z}-\frac{1}{z_0}\right)\left(1+\frac{1}{2z}+\frac{1}{2z_0}\right)-\frac{H}{a_0}(z_0-z)$$
$$-\frac{\mu}{2}\left(\frac{1}{z}-\frac{1}{z_0}\right)\left\{1+\tfrac{1}{4}(3+\mu)\left(\frac{1}{z}+\frac{1}{z_0}\right)\right\}+O\left(\frac{H}{a}\ln\frac{z_0}{z},\frac{e^2}{2},\frac{1}{3z^3}\right), \tag{39}$$

where suffix 0 denotes initial values. Since

$$\frac{r_p}{H} = \frac{a}{H}-z,$$

equation (39) may be rewritten as

$$\frac{r_{p0}-r_p}{H} = \tfrac{1}{2}\ln\frac{z_0}{z}+\frac{3}{8}\left(\frac{1}{z}-\frac{1}{z_0}\right)\left(1+\frac{1}{2z}+\frac{1}{2z_0}\right)-\frac{H}{a_0}(z_0-z)$$
$$-\frac{\mu}{2}\left(\frac{1}{z}-\frac{1}{z_0}\right)\left\{1+\tfrac{1}{4}(3+\mu)\left(\frac{1}{z}+\frac{1}{z_0}\right)\right\}+O\left(\frac{H}{a}\ln\frac{z_0}{z},\frac{e^2}{2},\frac{1}{3z^3}\right). \tag{40}$$

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On writing $z = ae/H$ and eliminating a/a_0 by means of (39), we obtain the perigee distance in terms of eccentricity as

$$\frac{r_{p0}-r_p}{H} = \tfrac{1}{2}\ln\frac{e_0(1+e)}{e(1+e_0)} + \frac{3H}{8a_0}\left(\frac{1+e_0}{e}-\frac{1}{e_0}\right)\left\{1+\frac{H}{2a_0}\left(\frac{1}{e}+\frac{1}{e_0}\right)\right\}$$
$$-\frac{\mu H}{2a_0}\left(\frac{1+e_0}{e}-\frac{1}{e_0}\right)\left\{1+\frac{H}{4a_0}(3+\mu)\left(\frac{1}{e}+\frac{1}{e_0}\right)\right\}+O\left(\frac{H}{a}\ln\frac{e_0}{e}, \frac{e^2}{2}, \frac{H^3}{3a^3e^3}\right). \quad (41)$$

This equation may be written in the form

$$\frac{r_{p0}-r_p}{H} = \left(\frac{r_{p0}-r_p}{H}\right)_{\text{sph. atm.}} - \frac{\mu\xi}{z_0}, \quad (42)$$

where the suffix 'sph. atm.' denotes the expression obtained in part I (the terms in (41) independent of μ) and order terms have been omitted. From (41), ξ is given by

$$\xi = \frac{1}{2}\left\{\frac{(1+e_0)\,e_0}{e}-1\right\}\left\{1+\frac{1}{4z_0}(3+\mu)\left(\frac{e_0}{e}+1\right)\right\} \quad (43)$$

and is plotted against e/e_0 for various z_0 in figure 5, with the term $(1+e_0)$ replaced by $(1+e_0-e)$. This change is necessary in order to make ξ zero at $e = e_0$; the e term was dropped in equation (41) because it was smaller than the neglected $(H/a)\ln(e_0/e)$ term. Figure 5 is drawn for $H/a = 0{\cdot}008$; but the values for $H/a = 0{\cdot}006$ are so close as to be indistinguishable.

Figure 5 shows that the variations of ξ with z_0 and μ are small: the $\mu = 1$, $z_0 = 30$ curve would in most circumstances be accurate enough to represent ξ for all values of z_0 and μ, so that ξ could be written, with maximum error 10 %, as

$$\xi \simeq \frac{1}{50}\left(\frac{e_0}{e}-1\right)\left(31+\frac{e_0}{e}\right).$$

It is also worth noting that (38) may be rewritten as

$$\frac{1}{H}\frac{\mathrm{d}a}{\mathrm{d}(z+\mu)} = 1+\frac{1}{2(z+\mu)}+\frac{3}{8(z+\mu)^2}+\frac{3}{8(z+\mu)^3}-\frac{H}{a}+O\left(\frac{e}{z^2}, \frac{e^2}{z}, \frac{1}{z^4}, \frac{\mu}{z^4}, \frac{\mu^2}{z^3}, \frac{\mu^3}{z^4}\right).$$

This means that, when ϕ_p is constant, the spherical-atmosphere results still hold, to the order of accuracy indicated, if z is replaced by $(z+\mu)$, that is, if e is replaced by $(e+\mu H/a_0)$. This representation can be useful—e.g. figure 4 of part I is valid for an atmosphere with day-to-night variation if e is read as $(e+\mu H/a_0)$—but it is slightly less accurate because the μ^2/z^3 term in the equation above gives rise to a H^2/a^2e^2 term in the equation for $(r_{p0}-r_p)/H$.

The orbital period T is given in terms of the eccentricity by the relation

$$\frac{T}{T_0} = \left(\frac{a}{a_0}\right)^{\frac{3}{2}} = \left(\frac{1-e_0}{1-e}\right)^{\frac{3}{2}}\left(\frac{r_p}{r_{p0}}\right)^{\frac{3}{2}}. \quad (44)$$

On substituting for r_p/r_{p0} from (41), we have

$$\frac{T}{T_0} = \left(\frac{1-e_0}{1-e}\right)^{\frac{3}{2}}\left[\!\left[1-\frac{3H}{4r_{p0}}\left[\ln\frac{e_0(1+e)}{e(1+e_0)}+\frac{3H}{4a_0}\left(\frac{1+e_0}{e}-\frac{1}{e_0}\right)\left\{1+\frac{H}{2a_0}\left(\frac{1}{e}+\frac{1}{e_0}\right)\right\}\right.\right.\right.$$
$$\left.\left.\left.-\frac{\mu H}{a_0}\left(\frac{1+e_0}{e}-\frac{1}{e_0}\right)\left\{1+\frac{H}{4a_0}(3+\mu)\left(\frac{1}{e}+\frac{1}{e_0}\right)\right\}\right]+O\left(\frac{H^2}{a^2}\ln\frac{e_0}{e}, \frac{He^2}{2a}, \frac{H^4}{3a^4e^3}\right)\right]\!\right] \quad (45)$$

$$= \left(\frac{T}{T_0}\right)_{\text{sph. atm.}}\left\{1+\frac{3H}{2r_{p0}}\frac{\mu\xi}{z_0}\right\}. \quad (46)$$

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5·2·2. *Eccentricity in terms of time*

We next determine the variation of eccentricity with time. Equation (36) can, with the aid of (33), be rewritten as

$$\Delta x = -2\pi\delta a^2 \rho_p \exp(-z)\, I_1(z) \left[1+2e-\mu\left(\frac{1}{2z}-\frac{3}{8z^2}\right)+O\left(e^2, \frac{H}{a}, \frac{3\mu}{8z^3}\right)\right]. \tag{47}$$

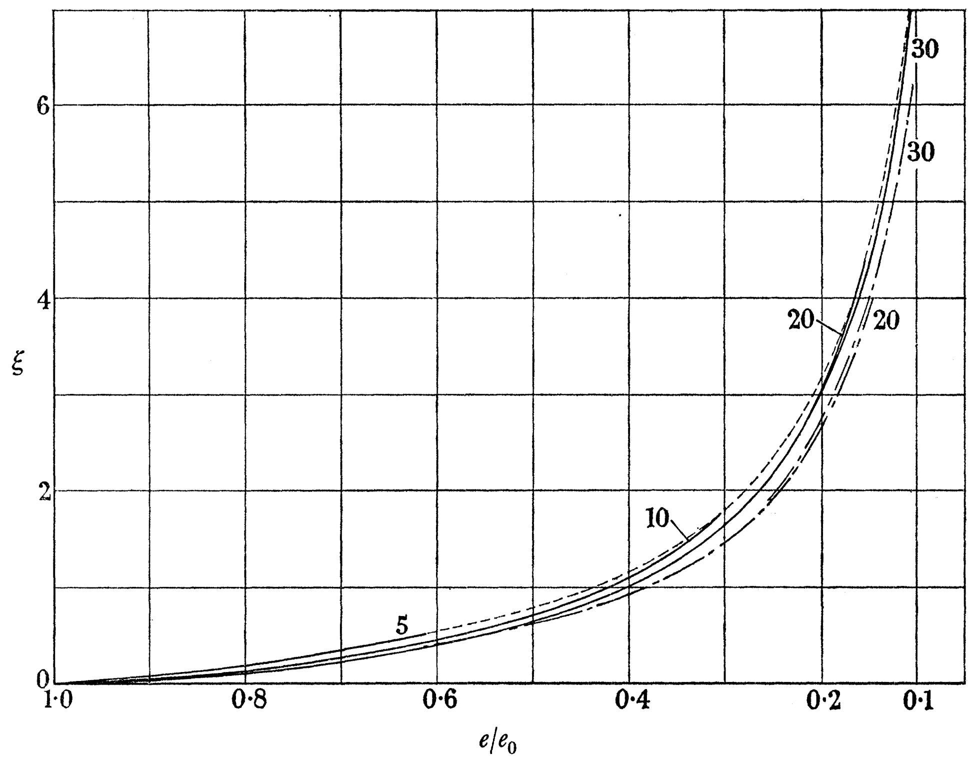

FIGURE 5. Variation of ξ with e/e_0. Numbers on curves are values of z_0.
——, $\mu = 1$; — — —, $\mu = -1$; -----, locus of end-points, $\mu = 1$.

To eliminate $\rho_p \exp(-z)$ from (47), we observe that, for ϕ_p constant, (37) gives

$$\rho_p \exp(-z) = \rho_{p0} \exp\{\beta(a_0-a)-z_0\}, \tag{48}$$

while

$$\beta(a_0-a) = \ln\frac{x_0 I_1(z_0)}{x I_1(z)} - \frac{x_0-x}{a_0} - \frac{\mu\xi}{z_0} + O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2 e}\right) \tag{49}$$

from equation (28) of part I. On using these last two equations to eliminate $\rho_p \exp(-z)$ from (47) and noting that $\Delta t = T = T_0(a/a_0)^{\frac{3}{2}}$, we obtain

$$\frac{\Delta x}{\Delta t} = -\frac{Ba_0^2}{x}\left(\frac{a}{a_0}\right)^{\frac{1}{2}}\left(\exp\frac{x}{a_0}\right)\left\{1-\frac{\mu\xi}{z_0}+\frac{\mu^2\xi^2}{2z_0^2}\right\}\left\{1+2e-\mu\left(\frac{1}{2z}-\frac{3}{8z^2}\right)+O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2 e}, \frac{\mu}{2z^3}\right)\right\}, \tag{50}$$

where

$$B = \frac{2\pi}{T_0}\delta x_0 \rho_{p0} I_1(z_0) \exp(-z_0-e_0) \tag{51}$$

and the $O(3\mu/8z^3)$ term which occurs has been ‘rounded up’ to $O(\mu/2z^3)$. On multiplying

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out the expressions in braces in (50), expressing ξ in terms of z by (42) and (40), and expanding $(a/a_0)^{\frac{1}{2}}$ by means of (39), we find

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{Ba_0^2}{x}\left(\exp\frac{x}{a_0}\right)\left\{1-\tfrac{1}{2}e_0+\tfrac{5}{2}e-\mu\left(\frac{1-\frac{1}{2}e_0}{z}-\frac{1}{2z_0}-\frac{3}{8z_0^2}\right)+\frac{\mu^2}{4}\left(\frac{1}{z}-\frac{1}{z_0}\right)^2 + O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2e}, \frac{\mu}{2z^3}\right)\right\}. \quad (52)$$

On inverting equation (52), expanding the exponential, and writing $z = x/H$, we have

$$-Ba_0^2\frac{\mathrm{d}t}{\mathrm{d}x} = x\left[1+\tfrac{1}{2}e_0-\frac{7x}{2a_0}+\mu\left\{(1+\tfrac{1}{2}e_0)\frac{H}{x}-\frac{H}{2x_0}-\frac{3H^2}{8x_0^2}\right\}+\mu^2\left\{\frac{3H^2}{4x^2}-\frac{H^2}{2xx_0}\right\} + O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2e}, \frac{\mu}{2z^3}\right)\right]. \quad (53)$$

On integrating from x_0 to x, and writing $e/e_0 = \lambda$, we obtain

$$\frac{2B}{e_0^2}t = 1-\lambda^2+\frac{e_0}{6}(2\lambda^3+9\lambda^2-11)+\frac{\mu}{2z_0}\left\{3-4\lambda+\lambda^2-\frac{3}{4z_0}(1-\lambda^2)\right\}+O\left(e^2, \frac{H}{a}\ln\frac{1}{\lambda}, \frac{H^2}{a^2e}, \frac{\mu}{2z^3}\right), \quad (54)$$

after omitting a term $(\mu H/a_0)(1-\lambda)(1+2\lambda-\lambda^2)$, which is of order H/a, and a term $(\mu^2/2z_0^2)\{3\ln(1/\lambda)-2(1-\lambda)\}$, which has a maximum value of $0{\cdot}015\mu^2$ (when $z = 3$ and $z_0 = 5{\cdot}7$) and can be written $O(0{\cdot}4\mu^2/z^3)$, which is less than the last of the existing order terms. Let t_L denote the value of t given by (54) without the order terms when $\lambda = 0$, so that

$$t_L = \frac{e_0^2}{2B}\left\{1-\tfrac{11}{6}e_0+\frac{3\mu}{2z_0}\left(1-\frac{1}{4z_0}\right)\right\} \quad (55)$$

and t_L is approximately the satellite's lifetime. On dividing (54) by (55) we find

$$\frac{t}{t_L} = 1-\lambda^2-\frac{e_0\lambda^2}{3}(1-\lambda)-\frac{2\mu\lambda}{z_0}(1-\lambda)+O\left(e^2, \frac{H}{a}\ln\frac{1}{\lambda}, \frac{H^2}{a^2e}, \frac{\mu}{2z^3}, \frac{\mu^2}{3z^2}\right), \quad (56)$$

where the term $3\mu^2\lambda(1-\lambda)/z_0^2$ has been written as $O(\mu^2/3z^2)$.

Equation (56) gives t in terms of $\lambda(=e/e_0)$. To obtain e in terms of t is straightforward, but it should be emphasized that the error terms are bound to become large as $t\to t_L$. For an equation of the form

$$t/t_L = 1-\lambda^2+O(\epsilon),$$

where ϵ is small, may be rewritten

$$\lambda = \sqrt{\left(1-\frac{t}{t_L}\right)}+O\left(\frac{\epsilon}{2\lambda}\right) \quad (57)$$

and, as $\lambda\to 0$, $O(\epsilon/2\lambda)$ becomes large. On expressing λ in terms of t from (56) and altering the order terms in accordance with (57), we have

$$\lambda = \frac{e}{e_0} = \left(1-\frac{t}{t_L}\right)^{\frac{1}{2}}\left[1-\frac{e_0}{6}\left\{1-\left(1-\frac{t}{t_L}\right)^{\frac{1}{2}}\right\}+O(e_0^2)\right]-\frac{\mu}{z_0}\left\{1-\left(1-\frac{t}{t_L}\right)^{\frac{1}{2}}\right\} + O\left(\frac{e_0}{2z}\ln\frac{e_0}{e}, \frac{e_0}{2z^2}, \frac{\mu e_0}{2ez^3}, \frac{\mu^2 e_0}{6ez^2}\right). \quad (58)$$

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Figure 6 shows how e/e_0 varies with t/t_L for various values of μ/z_0, as given by equation (56), with e_0 taken as 0 because it has already been established that the effect of e_0 and e_0^2 terms is not significant (King-Hele 1960).

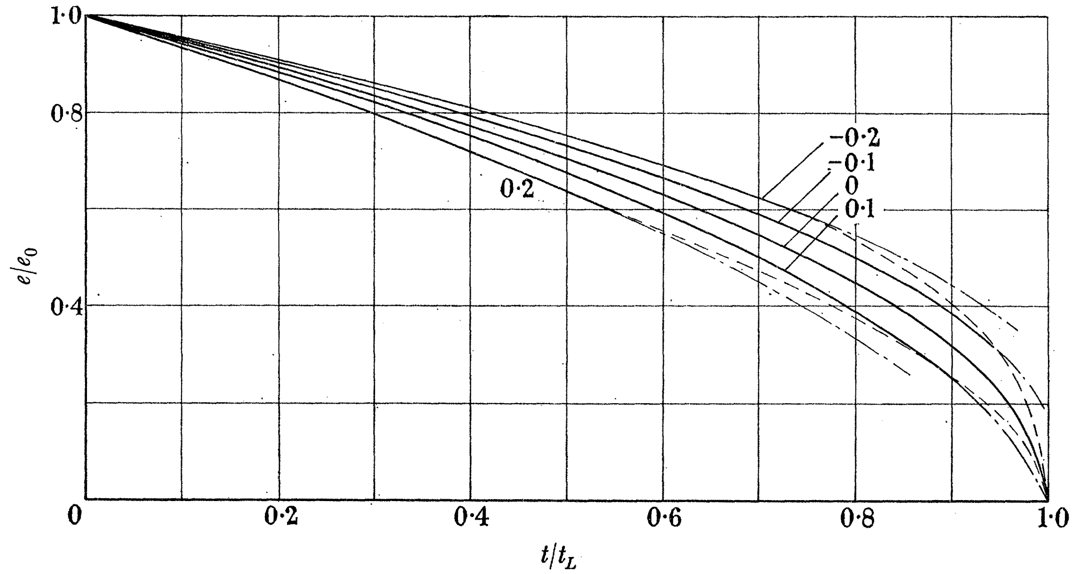

FIGURE 6. Variation of eccentricity e with time t for phase 1 (constant ϕ_p). Numbers on curves are values of μ/z_0. –·–·–, Boundary of validity for phase 1 ($e/e_0 = 3/z_0$, $|\mu| = 1$); – – –, curves continued beyond boundary of validity for interpolation purposes.

It is of interest that when the e_0 term in (56) is ignored (a legitimate simplification, since it never exceeds $e_0/20$), equation (56) may be rewritten as

$$\left(\frac{e+\mu H/a_0}{e_0+\mu H/a_0}\right)^2 = 1-\frac{t}{t_L}+O\left(e^2, \frac{H}{a}\ln\frac{1}{\lambda}, \frac{H^2}{a^2 e}, \frac{\mu}{2z^3}, \frac{\mu^2}{3z^2}, \frac{\mu^2}{z_0^2}\right). \tag{59}$$

Thus, to our order of accuracy, $(e+\mu H/a_0)^2$ varies linearly with time, and the equation has the same form as for a spherical atmosphere, with e replaced by $(e+\mu H/a_0)$.

5·2·3. *Lifetime in terms of $\dot{T}$*

We next find an expression for the satellite's 'lifetime', t_L, in terms of $\dot{T}$. Since $\Delta t = T$, we have

$$\dot{T} = \frac{\Delta T}{T} = \frac{3\Delta a}{2a}, \tag{60}$$

while from (31) we obtain, on using (33) and (37),

$$\Delta a = -2\pi\delta a^2\rho_p \exp(-z)\, I_1(z)\left[1+2e+\frac{1}{2z}+\frac{3}{8z^2}-\mu\left(\frac{1}{2z}+\frac{3}{8z^2}\right)+O\left(e^2, \frac{H}{a}, \frac{3}{8z^3}\right)\right]. \tag{61}$$

On substituting (61) into (60), taking initial values and introducing B by means of (51), we have

$$\dot{T}_0 = -\frac{3BT_0}{2e_0}\left\{1+3e_0+\frac{1}{2z_0}+\frac{3}{8z_0^2}-\mu\left(\frac{1}{2z_0}+\frac{3}{8z_0^2}\right)+O\left(e^2, \frac{H}{a}, \frac{3}{8z_0^3}\right)\right\}. \tag{62}$$

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Finally, multiplying (62) by (55) gives

$$t_L = -\frac{3e_0 T_0}{4\dot{T}_0}\left\{1+\tfrac{7}{6}e_0+\frac{1}{2z_0}+\frac{3}{8z_0^2}+\frac{\mu}{z_0}\left(1-\frac{3\mu}{4z_0}\right)+O\left(e^2, \frac{H}{a}, \frac{3}{8z_0^3}\right)\right\}. \tag{63}$$

$t_L(-\dot{T}_0/e_0 T_0)$ is plotted against e_0 in figure 7 for various H/a_0 and μ.

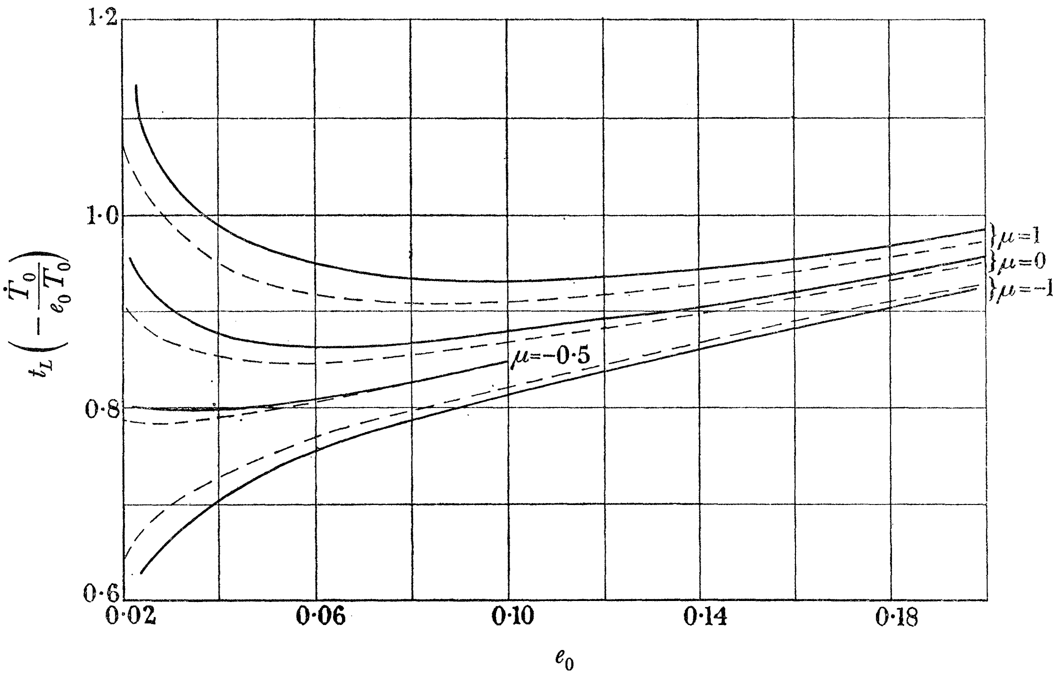

FIGURE 7. The effect of μ on lifetime t_L (phase 1, constant ϕ_p).
——, $H/a_0 = 0·008$; – – – –, $H/a_0 = 0·006$.

5·3. *Solution when ϕ_p is variable*

5·3·1. *Assumed variation of ϕ_p*

For most orbits the variation of ϕ_p with time is usually of a basically linear form. This is not always true if the orbital inclination i is large, since the possible forms of variation for ϕ_p can be complicated, as equation (19) suggests. If the orbital inclination is less than about 40°, however, the variation of ϕ_p with time is always basically linear (though with some slight departures from linearity), because the dominant factor is the west-to-east motion of the perigee point relative to the Sun (with its north–south excursions providing the departures from linearity). This basic linearity may be seen for example in figure 2 of the paper by King-Hele & Rees (1963): this shows the variation in Sun–perigee angle, which is of the same form as that of ϕ_p, for a typical satellite, Explorer 1 ($i = 33°$). In these conditions the first term in (19) is dominant, and, since ω, Ω and L all vary almost linearly with time, (19) may be written

$$\cos\phi_p \simeq \cos^2\tfrac{1}{2}\epsilon\cos^2\tfrac{1}{2}i\cos(P'+Q't)+\text{smaller terms},$$

where P' and Q' are constants. There are two possible approximations for ϕ_p. The first is to ignore the smaller terms in the above equation completely: since $\cos^2\frac{1}{2}\epsilon = 0·96$, this would mean that ϕ_p could never be less than 16° (or greater than 164°) even for equatorial orbits, whereas it can be as low as zero (or as great as 180°). The second possible approximation, which appears to be the better one, is to take

$$\cos\phi_p = \cos(P'+Q't). \tag{64}$$

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This allows ϕ_p to take its full range of values and also gives the linear variation of ϕ_p with t suggested in the early part of this paragraph.

Since e^2, and hence z^2, decreases almost linearly with time (Cook *et al.* 1960), the best simple approximation for t in terms of z, which we shall adopt, is a linear variation with z^2, giving

$$\cos\phi_p = \cos\left(P - \frac{Qz^2}{z_0^2}\right), \tag{65}$$

where P and Q are constants and Q is assumed positive. Q is positive if $i < 40°$. In the other conditions where a linear variation of ϕ_p with t is a particularly good approximation, namely when $140° < i < 180°$ or when i is near $97°$ and the orbital plane contains the Sun (as with Samos 2), the minus sign in (65) must be replaced by a plus. The corresponding changes in the results are indicated later.

In (65) P is the end-value of ϕ_p, when $z = 0$, and Q is the change in ϕ_p during the life of the satellite.

5·3·2. *Perigee distance and orbital period in terms of z*

To obtain the variation of r_p with z, we return to equation (38), which, on expressing μ in terms of F by (34) and expanding in powers of F, gives

$$\frac{1}{H}\frac{da}{dz} = 1 + \frac{1}{2z} + \frac{3}{8z^2} + \frac{3}{8z^3} - \frac{H}{a} - \frac{1}{2z^2}\left(1 + \frac{3}{2z}\right) F\cos\phi_p + \frac{1}{2z^2}\left(1 + \frac{1}{z}\right) F^2\cos^2\phi_p + O\left(\frac{e}{z^2}, \frac{He}{a}, \frac{1}{z^4}, \frac{F^3\cos^3\phi_p}{2z^2}\right). \tag{66}$$

The maximum value of the F^3 order term will always be less than the maximum value of $1/z^4$ if $F\cos\phi_p < 0{\cdot}6$, so we have introduced no significant new error term, provided F remains less than 0·6. On substituting (65) into (66), we find

$$\frac{1}{H}\frac{da}{dz} = 1 + \frac{1}{2z} + \frac{3}{8z^2} + \frac{3}{8z^3} - \frac{H}{a} - \frac{F}{2z^2}\left(1 + \frac{3}{2z}\right)\left(\cos P\cos\frac{Qz^2}{z_0^2} + \sin P\sin\frac{Qz^2}{z_0^2}\right) + \frac{F^2}{4z^2}\left(1 + \frac{1}{z}\right)\left(1 + \cos 2P\cos\frac{2Qz^2}{z_0^2} + \sin 2P\sin\frac{2Qz^2}{z_0^2}\right) + O\left(\frac{e}{z^2}, \frac{He}{a}, \frac{1}{z^4}, \frac{F^3}{2z^2}\right). \tag{67}$$

To integrate (67) we make use of the Fresnel integrals, C and S, defined by

$$C(u) = \int_0^u \cos\frac{\pi t^2}{2}\,dt, \quad S(u) = \int_0^u \sin\frac{\pi t^2}{2}\,dt, \tag{68}$$

where in our analysis the variable u will be defined by the equations

$$u_0 = \left(\frac{2Q}{\pi}\right)^{\frac{1}{2}}, \quad \frac{u}{u_0} = \frac{z}{z_0}. \tag{69}$$

The Fresnel integrals have been tabulated in terms of the argument u by Jahnke, Emde & Lösch (1960) and more fully by Van Wijngaarden & Scheen (1949). In integrating (67) we also introduce the sine and cosine integrals, Ci and Si, defined by

$$\mathrm{Si}\,(\nu) = \int_0^\nu \frac{\sin y}{y}\,dy, \quad \mathrm{Ci}\,(\nu) = -\int_\nu^\infty \frac{\cos y}{y}\,dy \tag{70}$$

and tabulated by Jahnke *et al.* (1960) and in more detail by the National Bureau of Standards (1940, 1954).

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On integrating (67) between the limits z and z_0, assuming F is constant, and utilizing (68)–(70), we obtain after some reduction

$$\frac{a_0-a}{H} = \left(\frac{a_0-a}{H}\right)_{\text{sph. atm.}} + \Psi + O\left(\frac{F^3}{2z}\right), \tag{71}$$

where the first term on the right is given by the terms independent of μ in equation (39), and

$$\begin{aligned}\frac{2z_0\Psi}{F} = {} & \left(1+\frac{3}{4z_0}\right)\cos\phi_{p0} - \frac{z_0}{z}\left(1+\frac{3}{4z}\right)\cos\phi_p + \pi u_0[\{S(u_0)-S(u)\}\cos P - \{C(u_0)-C(u)\}\sin P] \\ & + \frac{3Q}{4z_0}\left[\left\{\mathrm{Si}\,(Q) - \mathrm{Si}\left(\frac{Qz^2}{z_0^2}\right)\right\}\cos P - \left\{\mathrm{Ci}\,(Q) - \mathrm{Ci}\left(\frac{Qz^2}{z_0^2}\right)\right\}\sin P\right] \\ & - F\left[\left(1+\frac{1}{2z_0}\right)\cos^2\phi_{p0} - \frac{z_0}{z}\left(1+\frac{1}{2z}\right)\cos^2\phi_p\right] \\ & - \frac{F}{2}\Bigg[\pi u_0\surd 2[\{S(u_0\surd 2)-S(u\surd 2)\}\cos 2P - \{C(u_0\surd 2)-C(u\surd 2)\}\sin 2P] \\ & + \frac{Q}{z_0}\left[\left\{\mathrm{Si}\,(2Q) - \mathrm{Si}\left(\frac{2Qz^2}{z_0^2}\right)\right\}\cos 2P - \left\{\mathrm{Ci}\,(2Q) - \mathrm{Ci}\left(\frac{2Qz^2}{z_0^2}\right)\right\}\sin 2P\right]\Bigg].\end{aligned} \tag{72}$$

Equation (72) applies for $i < 40°$. If $140° < i < 180°$ (or $i \simeq 97°$ and the Sun is in the orbital plane), and ϕ_p is taken as $P + Qz^2/z_0^2$, as mentioned in the sentence after equation (65), equation (72) is still valid if the four minus signs between the eight terms in braces are replaced by plus signs.

On subtracting $(z_0 - z)$ from both sides of (71), we obtain a corresponding equation for perigee distance

$$\frac{r_{p0}-r_p}{H} = \left(\frac{r_{p0}-r_p}{H}\right)_{\text{sph. atm.}} + \Psi + O\left(\frac{F^3}{2z}\right), \tag{73}$$

where the spherical-atmosphere value is given by the terms independent of μ in (40) or (41). Equation (72) gives Ψ in terms of u and z: to express it in terms of e we should use the equation

$$\frac{u}{u_0} = \frac{z}{z_0} = \frac{e}{e_0}\{1 - e_0 + e + O(e^2)\}. \tag{74}$$

Ψ depends on too many parameters for graphical presentation to be feasible: the leading term, however, is usually

$$\frac{F}{2}\left(\frac{\cos\phi_{p0}}{z_0} - \frac{\cos\phi_p}{z}\right),$$

while the leading term in $(r_{p0}-r_p)/H$ is $\frac{1}{2}\ln(z_0/z)$. Thus if $z_0/z = 10$ and $z = 3$, the leading term is of order 1 and the correction is of order $\frac{1}{6}F\cos\phi_p$.

The orbital period T can, by the same process as in equations (44) to (46), be expressed in the form

$$\frac{T}{T_0} = \left(\frac{T}{T_0}\right)_{\text{sph. atm.}} \left\{1 - \frac{3H\Psi}{2r_{p0}}\right\}. \tag{75}$$

5·3·3. *Eccentricity in terms of time*

We determine the variation of eccentricity with time only when F is small, $F < 0{\cdot}2$. When ϕ_p is variable, equation (47) may be rewritten, on substituting for ρ_p from (37), as

$$\Delta x = -2\pi\delta a^2\rho_0 \exp\{\beta(r_0-a)\}\, I_1(z)\left\{1 + 2e + F\cos\phi_p + O\left(e^2, \frac{H}{a}, Fe, \frac{F}{z}\right)\right\}. \tag{76}$$

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On substituting for $\exp(-\beta a)$ from (71) and again utilizing the equation $\Delta t = T_0(a/a_0)^{\frac{3}{2}}$, we have

$$\frac{\Delta x}{\Delta t} = -\frac{B'a_0^2}{x}\left(\frac{a}{a_0}\right)^{\frac{1}{2}}\left(\exp\frac{x}{a_0}\right)\left\{1+2e+F\cos\phi_p+O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2e}, Fe, \frac{F}{z}\right)\right\}, \tag{77}$$

where
$$B' = \frac{2\pi}{T_0}\delta\rho_0 x_0 I_1(z_0)\exp\{\beta(r_0-a_0)-e_0\} = \frac{B}{1+F\cos\phi_{p0}}. \tag{78}$$

Expanding $(a/a_0)^{\frac{1}{2}}$ as $\{1-\frac{1}{2}(e_0-e)\}$, and $\exp(x/a_0)$ as $(1+e)$, and inverting, equation (77) becomes

$$-B'a_0^2\frac{\mathrm{d}t}{\mathrm{d}x} = x\left\{1+\frac{x_0}{2a_0}-\frac{7x}{2a_0}-F\cos\left(P-\frac{Qz^2}{z_0^2}\right)+O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2e}, Fe, \frac{F}{z}\right)\right\}. \tag{79}$$

Integration of (79) between the limits x and x_0 gives

$$\frac{2B't}{e_0^2} = 1-\lambda^2+\frac{e_0}{6}(2\lambda^3+9\lambda^2-11)+\frac{F}{Q}\left\{\sin(P-Q)-\sin\left(P-\frac{Qx^2}{x_0^2}\right)\right\}$$
$$+O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2e}, Fe, \frac{F}{z}\right). \tag{80}$$

Let t_L denote the value of t given by (80) when $\lambda = 0$, so that

$$\frac{2B't_L}{e_0^2} = 1-\tfrac{11}{6}e_0+\frac{F}{Q}\{\sin(P-Q)-\sin P\}. \tag{81}$$

Dividing (80) by (81),

$$\frac{t}{t_L} = 1-\lambda^2-\frac{e_0\lambda^2}{3}(1-\lambda)+\frac{F}{Q}\{(1-\lambda^2)\sin P-\sin(P-Q\lambda^2)+\lambda^2\sin(P-Q)\}$$
$$+O\left(e^2, \frac{H}{a}\ln\frac{x_0}{x}, \frac{H^2}{a^2e}, Fe, \frac{F}{z}\right). \tag{82}$$

Equation (82) may be rewritten to give e in terms of t as

$$\frac{e}{e_0} = \left(1-\frac{t}{t_L}\right)^{\frac{1}{2}}\left[1-\frac{e_0}{6}\left\{1-\left(1-\frac{t}{t_L}\right)^{\frac{1}{2}}\right\}+O(e_0^2)\right]+\frac{F}{2Q(1-t/t_L)^{\frac{1}{2}}}$$
$$\times\left\{\frac{t}{t_L}\sin P-\sin\left(P-Q+Q\frac{t}{t_L}\right)+\left(1-\frac{t}{t_L}\right)\sin(P-Q)\right\}+O\left(\frac{e_0}{2z}\ln\frac{e_0}{e}, \frac{e_0}{2z^2}, \frac{Fe_0}{2}, \frac{Fz_0}{2z^2}\right). \tag{83}$$

It should be remembered that in these equations P represents the end-value of ϕ_p and $P-Q = \phi_{p0}$. e is plotted against t in figure 8 for various values of P and Q: the curves shown include the largest possible departures from the $F = 0$ curve.

5·3·4. *Lifetime in terms of* $\dot{T}$

To obtain an expression for the satellite's lifetime, we rewrite (61) as

$$\Delta a = -2\pi\delta a^2\rho_0\exp\{\beta(r_0-a)\}I_1(z)\left[1+2e+\frac{1}{2z}+\frac{3}{8z^2}+F\cos\phi_p+O\left(e^2, \frac{H}{a}, \frac{3}{8z^3}, Fe_0, \frac{F}{z}\right)\right], \tag{84}$$

whence, using (60) and (78),

$$\dot{T}_0 = -\frac{3B'T_0}{2e_0}\left\{1+3e_0+\frac{1}{2z_0}+\frac{3}{8z_0^2}+F\cos\phi_{p0}+O\left(e^2, \frac{H}{a}, \frac{3}{8z^3}, Fe_0, \frac{F}{z}\right)\right\}. \tag{85}$$

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Eliminating B' by means of (81), we have finally

$$t_L = -\frac{3e_0 T_0}{4\dot{T}_0}\left[1+\tfrac{7}{6}e_0+\frac{1}{2z_0}+\frac{3}{8z_0^2}+F\cos\phi_{p0}+\frac{F}{Q}\{\sin(P-Q)-\sin P\}+O\left(e_0^2, \frac{H}{a}, \frac{3}{8z_0^3}, Fe_0, \frac{F}{z_0}\right)\right]. \quad (86)$$

If ϕ_p completes many revolutions, as for satellites with long lifetimes, Q is large* and the F/Q term in (86) can be ignored. The formula then reduces to the spherical-atmosphere form, if $\dot{T}_0$ is replaced by $(\dot{T}_0)_{90} = \dot{T}_0/(1+F\cos\phi_{p0})$, i.e. $\dot{T}_0$ is 'corrected' to the value appropriate to a mean value of density, where $\phi = 90°$. In practice this means that the 'initial' $\dot{T}$ should be taken at $\phi_p = 90°$ or averaged over a cycle of ϕ_p. When Q takes small values, t_L is significantly affected by F, however.

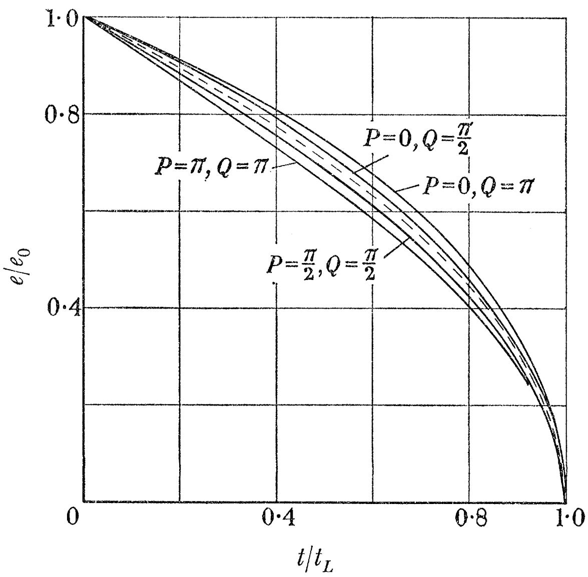

FIGURE 8. Variation of eccentricity e with time t for phase 1 (variable ϕ_p). ——, $F = 0{\cdot}2$; - - - - -, $F = 0$.

5·4. *Formula for air density in terms of $\dot{T}$*

The basic relation for determining air density at perigee, ρ_p, from the rate of change of orbital period, $\dot{T}$, is found by eliminating Δa between equations (35) and (60). We obtain

$$\rho_p = -\frac{\dot{T}}{3\delta}\left(\frac{2e}{\pi a H}\right)^{\frac{1}{2}}\left[1-2e+\tfrac{5}{2}e^2-\frac{1}{8z}-\frac{7}{128z^2}+\frac{5H}{4a}+\mu\left(\frac{1}{2z}+\frac{1}{16z^2}\right)+\frac{\mu^2}{4z^2}+O\left(e^3, \frac{1}{8z^3}\right)\right]. \quad (87)$$

The effect of the μ terms can be largely eliminated by expressing equation (87) in terms of H_1, where

$$H_1 = H\left(1-\frac{\mu}{z}\right) = H\left(1-\frac{\mu H}{x}\right). \quad (88)$$

Equation (87) then reduces to

$$\rho_p = -\frac{\dot{T}}{3\delta}\left(\frac{2e}{\pi a H_1}\right)^{\frac{1}{2}}\left[1-2e+\tfrac{5}{2}e^2-\frac{H_1}{8x}-\frac{7H_1^2}{128x^2}+\frac{5H_1}{4a}+\frac{\mu H_1}{a}-\frac{\mu^2 H_1^2}{8x^2}+O\left(e^3, \frac{1}{8z^3}\right)\right]. \quad (89)$$

* For example, for a satellite with a life of 50 years and $\dot{\phi}_p = 2$ deg/day, $Q = 200\pi$.

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The order terms in this equation have maximum magnitudes of 0·008 (when $e = 0·2$) and 0·005 (when $z = 3$). When $\mu = 1$, the μ terms have maximum magnitudes of about 0·008 and 0·014 (when $z = 3$); when $\mu = 0·6$ these values both reduce to 0·005. Thus if $\mu < 0·6$ the μ terms in (89) can be ignored and the equation is of the same form as for a spherically symmetrical atmosphere with H_1 instead of H. If $0·6 < \mu < 1$, the μ terms ought to be included but have only a small effect.

These relations are one-revolution results and apply for either variable or fixed ϕ_p.

6. Solution when $z \leqslant 3$ (phase 2)

If a satellite is in phase 1 when it first enters orbit, with an eccentricity of, say, 0·1, the time it spends in phase 2 (where e is less than about 0·02) will be much shorter than in phase 1, since e^2 varies almost linearly with time. Therefore ϕ_p is much more likely to be near-constant in phase 2 than in phase 1. Also, if phase 2 is short enough for ϕ_p to be near-constant, the perigee height is likely to be below 250 km, so that F is not likely to exceed 0·4.

We therefore concentrate most attention on the constant-ϕ_p results with $F < 0·4$.

6·1. *Solution when ϕ_p is constant*

6·1·1. *Treatment of equation for* da/dx

The important step in the solution is to rewrite the equation for da/dx in a directly integrable form. On dividing (31) by (32), we have

$$\frac{\mathrm{d}a}{\mathrm{d}x} = \frac{I_0 + I_1 F\cos\phi_p}{I_1 + \frac{1}{2}(I_0+I_2)\,F\cos\phi_p}\{1+O(e, eF\cos\phi_p)\}. \tag{90}$$

In phase 2, e is usually less than 0·02, and here e is when necessary assumed to be 0·02 in order terms. Terms of order e have not been retained in (90) because they are available from part I, while the terms in $eF\cos\phi_p$ have been disregarded because we are assuming $|F\cos\phi_p| < 0·4$.

If $|F\cos\phi_p| < 0·4$ equation (90) can be neatly integrated. We write

$$\frac{I_1 + \frac{1}{2}(I_0+I_2)\,F\cos\phi_p}{I_0 + I_1 F\cos\phi_p} \simeq \frac{I_1(z+\kappa)}{I_0(z+\kappa)}, \tag{91}$$

where κ is a quantity depending on $F\cos\phi_p$ but independent of z, and κ is chosen to make (91) as accurate as possible. When $\phi_p < 90°$, we may take

$$\kappa = F\cos\phi_p \quad (\phi_p < 90°). \tag{92}$$

The left-hand side of (91) then never differs from the right-hand side by a factor of more than 1·0199, that is $1+O(e)$, if $F\cos\phi_p < 0·4$ and the error is usually much less, as shown in figure 9 (a). When $\kappa < 0$, equation (92) still provides a good approximation, as shown by figure 9 (b), which is for the worst value of κ. It could be argued that the broken line in figure 9 (b) represents a good enough approximation to the full line, because the full line itself is slightly erroneous, neglecting as it does terms of $O(e)$ and $O(\kappa e)$. However, the broken line in figure 9 (b) must be regarded as an unsatisfactory approximation, because it crosses the z axis at a slightly incorrect value of z: consequently da/dx would tend to infinity at a slightly erroneous value of z. To avoid this difficulty when $\phi_p > 90°$ ($\kappa < 0$), we choose

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κ so that the left-hand and right-hand sides of (91) are zero at the same value of z. Since $I_1(z+\kappa)$ is zero when $z = -\kappa$, we require that

$$I_1(-\kappa)+\tfrac{1}{2}\{I_0(-\kappa)+I_2(-\kappa)\}F\cos\phi_p = 0 \quad (\phi_p > 90^\circ). \tag{93}$$

The solution of (93) can be written

$$\kappa = \frac{F\cos\phi_p}{1-0{\cdot}258\,F^2\cos^2\phi_p} \quad (\phi_p > 90^\circ), \tag{94}$$

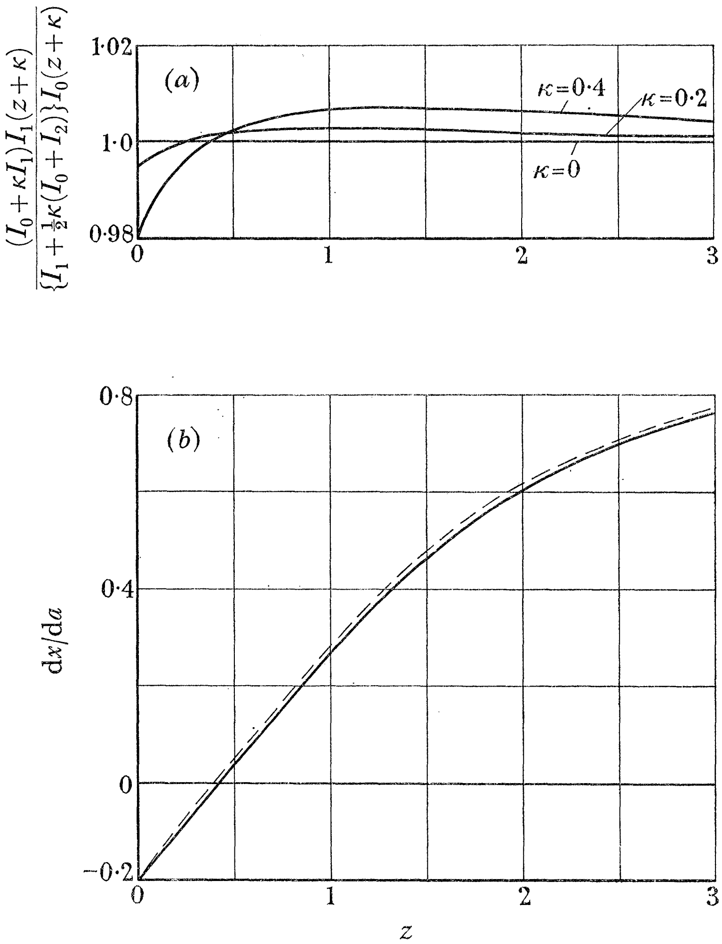

Figure 9. The accuracy of equation (91) with $\kappa = F\cos\phi_p$ when κ is (a) positive, and (b) negative. In (b), $\kappa = -0{\cdot}4$. ——, $\{I_1+\frac{1}{2}\kappa(I_0+I_2)\}/(I_0+\kappa I_1)$; – – – –, $I_1(z+\kappa)/I_0(z+\kappa)$.

with a maximum error in κ of 1 part in 3000. A more exact form, correct to 1 part in 10^4, is $\kappa = F\cos\phi_p(1+0{\cdot}249\kappa^2)$. Since $0{\cdot}249\kappa^2 < 0{\cdot}04$, κ is still very close to $F\cos\phi_p$. When the value of κ given by (94) is used, the maximum difference between the values of the left-hand and right-hand sides of (91) is 0·009, at $z \simeq 2$ and $\kappa = -0{\cdot}4$. Since the neglected $O(e)$ and $O(\kappa e)$ terms in (90) contribute as much error as this, the approximation (91) does not introduce any significant error. Thus we use (91), with κ given by (92) when $\phi_p < 90^\circ$ and by (94) when $\phi_p > 90^\circ$. For $\phi_p = 90^\circ$ the theory reduces to the spherical-atmosphere form.

We have so far assumed that ϕ_p remains constant, intending that the theory should cover the situation when ϕ_p varies little or not at all. However, if $\phi_p < 90^\circ$, z will decrease to zero before the end of the life, and at $z = 0$ the assumption of constant ϕ_p cannot be justified,

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because ϕ_p normally undergoes a discontinuous change at $z = 0$. For, as the orbit becomes circular, perigee becomes momentarily undefinable, and the new perigee which arises as soon as z departs from zero will in general appear at the point on the orbit where the density is lowest, so that the new 'post-circular' value of ϕ_p will be the nearest point on the orbit to the minimum-density point: ϕ_p will be between 90° and 180°, and its post-circular value will be independent of its pre-circular value (unless $\phi_p = 0$ just before z decreases to zero, for then the orbit must pass through the minimum-density point and the subsequent value of ϕ_p will always be 180°). Fortunately this discontinuity in ϕ_p does not affect the theory formally, because the lifetime after $z = 0$ turns out to be independent of ϕ_p to the order of accuracy which we are using. But the discontinuity must be kept in mind when interpreting the results.

6·1·2. *Perigee distance and orbital period in terms of z*

On substituting (91) into (90) we have

$$\frac{1}{H}\frac{\mathrm{d}a}{\mathrm{d}z} = \frac{I_0(z+\kappa)}{I_1(z+\kappa)}\{1+O(e,\kappa e)\}, \tag{95}$$

where κ is given by (92) when $\phi_p < 90°$ and by (94) when $\phi_p > 90°$. It is convenient to replace z by the variable ζ, defined by

$$\zeta = z+\kappa, \tag{96}$$

for, since κ is constant except for its discontinuity at $z = 0$, equation (95) may be rewritten

$$\frac{1}{H}\frac{\mathrm{d}a}{\mathrm{d}\zeta} = \frac{I_0(\zeta)}{I_1(\zeta)}\{1+O(e,\kappa e)\} \quad (z \neq 0). \tag{97}$$

Equation (97) can be integrated to give

$$\frac{a_1-a}{H} = \ln\left\{\frac{\zeta_1 I_1(\zeta_1)}{\zeta I_1(\zeta)}\right\}[1+O(e,\kappa e)] \quad (z \neq 0), \tag{98}$$

where a_1 and ζ_1 are the initial values of a and ζ. Hence the perigee distance r_p can be expressed as

$$\frac{r_{p1}-r_p}{H} = \ln\left\{\frac{\zeta_1 I_1(\zeta_1)}{\zeta I_1(\zeta)}\right\}[1+O(e,\kappa e)]-(\zeta_1-\zeta) \quad (z \neq 0). \tag{99}$$

Equations (98) and (99) are the same as for a spherically symmetrical atmosphere, except that ζ appears instead of z. Figure 10 shows the variation of perigee distance with ζ. This is the same as the corresponding diagram for a spherically symmetrical atmosphere, except that curves have been added for negative values of ζ. When $\zeta_1 \neq 3$, values of $(r_{p1}-r_p)$ can be obtained from $\{r_p(3)-r_p(\zeta)\}-\{r_p(3)-r_{p1}\}$, i.e. the difference in the ordinates at the appropriate values of ζ.

The end of the satellite's life occurs just before $(r_{p1}-r_p)$ tends to infinity, i.e. just before $\zeta \to 0$, or $z+\kappa \to 0$. If κ is negative ($\phi_p > 90°$), this means that at the end of the life z tends towards $-\kappa$, either from above or below, according as z_1 is greater or less than $|\kappa|$, and z never takes the value zero (unless it is zero initially): thus equation (99) and figure 17 give the complete solution for perigee distance. If κ is positive however ($\phi_p < 90°$), z will decrease to zero before the end of the life. While z is decreasing from z_1 to zero the value of r_p can be found in the normal way from (99), and the value of r_p at $z = 0$, r_{pz} say, is given

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by writing $\zeta = \kappa$ in (99). To evaluate the change in r_p after $z = 0$, the 'post-circular' phase, the circular orbit should be treated as a new initial condition, and the post-circular value of κ (always negative) should be used. If $\phi_p = 0$ in the pre-circular phase, the post-circular value of ϕ_p will be 180° and the post-circular value of κ is $-F/(1-0{\cdot}258F^2)$: otherwise the post-circular value of κ cannot be specified beforehand. Figure 11 shows the variation of r_p with z for an orbit with $z_1 = 3$ and $F = 0{\cdot}2$ when ϕ_p is either 0 or 180° initially. When

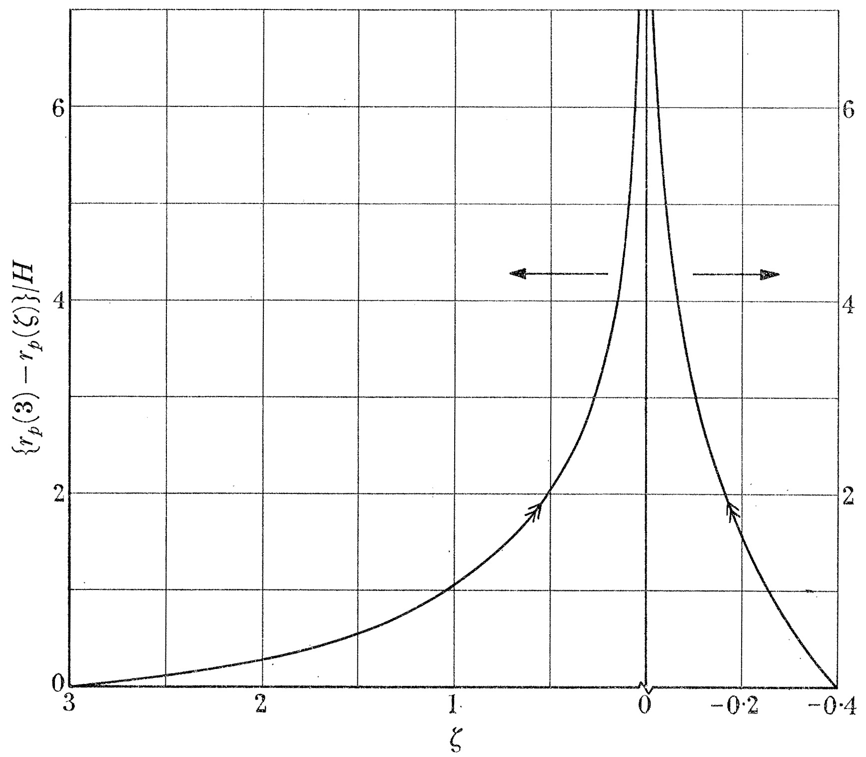

FIGURE 10. Variation of perigee distance r_p with ζ for phase 2 (constant ϕ_p). Double arrowheads on the curves indicate the direction of time increasing.

$\phi_p = 180°$, z steadily decreases towards its end-value near 0·2 (corresponding to $e = 0{\cdot}001$ if $H/a = 0{\cdot}005$); when $\phi_p = 0$, z decreases to zero and then in the post-circular phase $\phi_p = 180°$ so that $\kappa \simeq -0{\cdot}2$ and z increases towards a value near 0·2.

From equation (98) the orbital period T is given by

$$\frac{T}{T_1} = \left(\frac{a}{a_1}\right)^{\frac{3}{2}} = 1 - \frac{3H}{2a_1}\ln\left\{\frac{\zeta_1 I_1(\zeta_1)}{\zeta I_1(\zeta)}\right\} + O\left(\frac{eH}{a}, \frac{\kappa e H}{a}\right). \tag{100}$$

6·1·3. *The parameter ζ in terms of time*

We now find how eccentricity varies with time, by establishing the time variation of ζ, which is related to e by the equation

$$e = (H/a)\,(\zeta - \kappa). \tag{101}$$

The starting point is equation (32). In (32) we first write

$$I_1 + \tfrac{1}{2}(I_0 + I_2)\,F\cos\phi_p = I_1(\zeta)\,\{1 + O(\tfrac{1}{2}\kappa^2)\}, \tag{102}$$

where $\zeta = z + \kappa$. The term $O(\frac{1}{2}\kappa^2)$ is indicated by the Taylor expansion of $I_1(z+\kappa)$ and has also been verified numerically. Then, on replacing ρ_0 and r_0 in (32) by ρ_1 and r_1, taking r_1

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as the initial perigee distance, putting $\Delta t = T = T_1(a/a_1)^{\frac{3}{2}}$ and eliminating a by means of (98), we can write equation (32) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{B_3 a_1^2}{\zeta}\{1+O(e, \tfrac{1}{2}\kappa^2)\}, \tag{103}$$

where

$$B_3 = \frac{2\pi\delta\rho_1\zeta_1}{T_1} I_1(\zeta_1)\exp(-z_1). \tag{104}$$

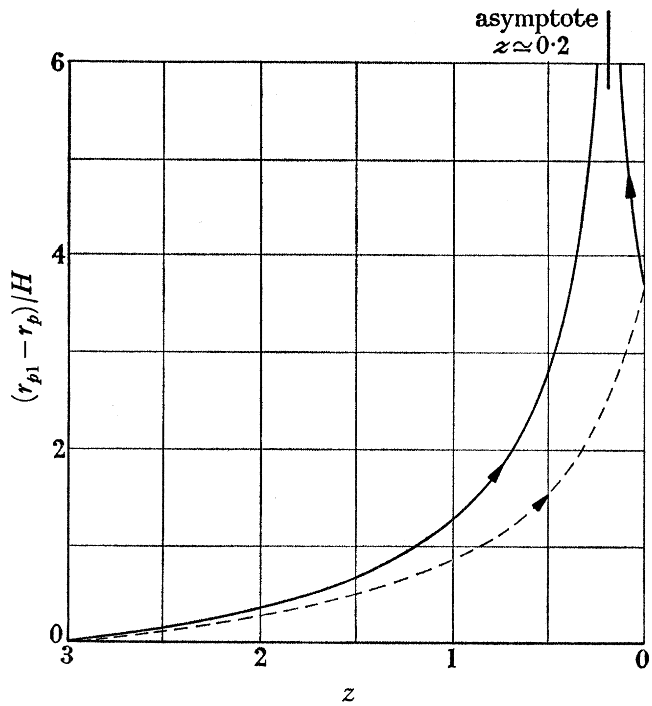

Figure 11. Variation of perigee distance r_p with z for an orbit with $z_1 = 3$ and $F = 0{\cdot}2$, when ϕ_p is constant. Arrows indicate direction of time increasing. ——, $\phi_p = 180°$; – – – –, $\phi_p = 0$.

If we again exclude the point $z = 0$, where κ suffers a discontinuity, (103) may be rewritten as

$$\frac{\mathrm{d}t}{\mathrm{d}\zeta} = -\frac{H\zeta}{a_1^2 B_3}\{1+O(e, \tfrac{1}{2}\kappa^2)\} \quad (z \neq 0). \tag{105}$$

Equation (105) does not adequately cover the situation when $\zeta_1 = 0$ $(z_1 = -\kappa)$. For, as $\zeta_1 \to 0$, B_3 becomes proportional to ζ_1^2, so that, by (105), $\mathrm{d}\zeta/\mathrm{d}t = 0$ when $\zeta_1 = 0$; thus the orbit contracts with $\zeta = 0$ $(z = -\kappa)$ and equation (105) becomes trivial. Discussion of this special case is deferred till § 6·1·5.

Excluding the special cases already mentioned, we may integrate (105) to give

$$\tau = \frac{H}{2a_1^2 B_3}(\zeta_1^2 - \zeta^2)\{1+O(e, \tfrac{1}{2}\kappa^2)\} \quad (z \neq 0,\ \zeta_1 \neq 0), \tag{106}$$

where $\tau = t - t_1$.

The next step is to find the satellite's lifetime. In parts I–IV the lifetime was defined as the value of τ when $e = 0$: this definition must now be generalized, and the obvious definition for the lifetime τ_L is the value of τ given by (106) when $\zeta = 0$. Thus we have

$$\tau_L \doteq H\zeta_1^2/2a_1^2 B_3. \tag{107}$$

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Dividing (106) by (107) gives

$$\frac{\tau}{\tau_L}\{1+O(e, \tfrac{1}{2}\kappa^2)\} = 1-\frac{\zeta^2}{\zeta_1^2} \quad (z \neq 0, \zeta_1 \neq 0). \tag{108}$$

This equation is of the same form as for a spherically symmetrical atmosphere, the only difference being that e (or z) is replaced by ζ.

If $\kappa < 0$, or if $\kappa > 0$ and $z > 0$, equation (108) can be used quite straightforwardly. If z reaches zero, however, it is not obvious that equation (107) gives a realistic estimate of the lifetime, since (106) has been forcibly carried across its discontinuity in obtaining (107). A little further investigation is needed, as follows.

Consider an orbit for which $z_1 = 0$ initially or, strictly, for which z_1 is nearly zero but sufficiently different from zero to have allowed a recognizable perigee position to develop, so that its post-circular value of κ is determinate. ζ then increases from an initial value κ (negative, since $\phi_p > 90°$) towards zero, and equations (105) and (106) are valid. When $z_1 \to 0$, so that $\zeta_1 \to \kappa$, the parameter

$$B_3 \to \frac{\pi\delta\rho_1\kappa^2}{T_1}\left\{1+O\left(\frac{\kappa^2}{8}\right)\right\},$$

on replacing the Bessel function I_1 in (104) by its series expansion. Thus for an initially circular orbit with $\zeta_1 = \kappa$, (107) gives

$$\tau_L = \frac{HT_1}{2\pi\delta\rho_1 a_1^2}\left\{1+O\left(\frac{\kappa^2}{8}\right)\right\}. \tag{109}$$

Equation (109) shows that the lifetime in an initially circular orbit is independent of κ (i.e. independent of ϕ_p) to the accuracy to which we are working, and is the same as for a spherically symmetrical atmosphere. This important result shows that the lifetime after $z = 0$ will be the same for any value of κ (less than 0·4). Therefore the lifetime obtained by putting $\zeta = 0$ in (106), even though it embodies the incorrect assumption that κ retains the same value to the end of the life, is still correct, because the lifetime after $z = 0$ is independent of κ and the use of a wrong value of κ is immaterial.

So equation (108) can still be used for the post-circular phase if $z = 0$ is treated as a new starting point. ζ is then calculated using the post-circular value of κ, and τ/τ_L is to be taken as $(\tau-\tau_Z)/(\tau_L-\tau_Z)$, where τ_Z is the time at $z = 0$. From (108), τ_Z is given by

$$\tau_Z = \tau_L\{1-\kappa^2/(z_1+\kappa)^2\}, \tag{110}$$

with κ having its pre-circular value.

Although the results are best given in the generalized form (108), it is also useful to show directly how τ/τ_L depends on κ. On expansion in powers of κ, equation (108) yields

$$\frac{\tau}{\tau_L} = \left\{1-\frac{e^2}{e_1^2}-\frac{2\kappa e}{z_1 e_1}\left(1-\frac{e}{e_1}\right)\right\}\left\{1+O\left(e, \frac{\kappa^2}{2}, \frac{\kappa^2}{z_1^2}\right)\right\}. \tag{111}$$

In practice equation (111) would be used only for $z_1 > 1$, since the error term $O(\kappa^2/z_1^2)$ becomes large when $z_1 < 1$. Equation (111) is, to the accuracy to which it has been carried, the same as the phase 1 result, equation (56). Therefore figure 6 can also be used in phase 2, if e/e_0 is read as e/e_1, t/t_L as τ/τ_L, and the numbers on the curves are read as values of κ/z_1. Equation (111), like (56), may be rewritten to give e in terms of time as

$$\frac{e}{e_1} = \left(1-\frac{\tau}{\tau_L}\right)^{\frac{1}{2}}\{1+O(e_1)\}-\frac{\kappa}{z_1}\left\{1-\left(1-\frac{\tau}{\tau_L}\right)^{\frac{1}{2}}\right\}+O\left(\frac{e_1\kappa^2}{4e}, \frac{\kappa^2}{2zz_1}\right). \tag{112}$$

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6·1·4. *Lifetime in terms of* $\dot{T}_1$

We next derive expressions for the lifetime τ_L in terms of $\dot{T}_1$. From (60) and (31) we have

$$\dot{T}_1 = -3\pi\delta a_1\rho_1 \exp\{\beta(r_1-a_1)\}[I_0(z_1)+I_1(z_1)F\cos\phi_p+O(e_1,\kappa e_1)]. \tag{113}$$

On writing
$$I_0(z_1)+I_1(z_1)F\cos\phi_p = I_0(z_1+\kappa)\{1+O(\tfrac{3}{8}\kappa^2)\}, \tag{114}$$

where the $O(\frac{3}{8}\kappa^2)$ term is deduced from the Taylor expansion of $I_0(z_1+\kappa)$, equation (113) becomes
$$\dot{T}_1 = -3\pi\delta a_1\rho_1 \exp(-z_1)\, I_0(\zeta_1)\{1+O(e_1,\tfrac{3}{8}\kappa^2)\}, \tag{115}$$

where $\zeta_1 = z_1+\kappa$, as before. Dividing (115) by (104) we have

$$\dot{T}_1 = -\frac{3B_3T_1a_1}{2\zeta_1}\frac{I_0(\zeta_1)}{I_1(\zeta_1)}\{1+O(e_1,\tfrac{3}{8}\kappa^2)\}. \tag{116}$$

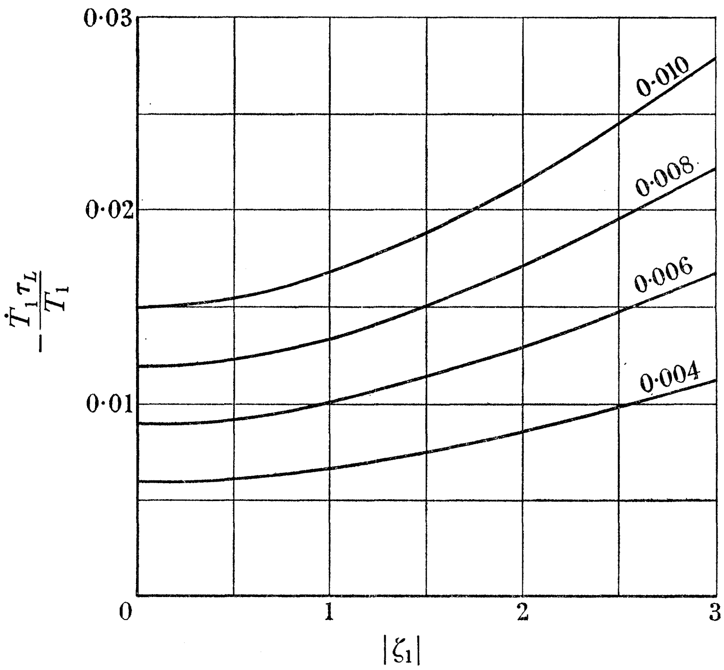

FIGURE 12. Variation of lifetime τ_L with ζ_1 for phase 2 (constant ϕ_p). Numbers on curves are values of H/a_1.

Multiplying (116) by (107), we obtain

$$\tau_L = -\frac{3T_1}{4\dot{T}_1}\frac{H}{a_1}\frac{\zeta_1 I_0(\zeta_1)}{I_1(\zeta_1)}\{1+O(e_1,\tfrac{3}{8}\kappa^2)\}. \tag{117}$$

This is the equation for lifetime: it is the same as for a spherically symmetrical atmosphere except that z_1 has been replaced by ζ_1. $\tau_L\dot{T}_1/T_1$, as given by (117), is plotted against ζ_1 for various H/a_1 in figure 12. Since $\zeta_1 = \kappa$ when $z_1 = 0$, figure 12 confirms that for an initially circular orbit τ_L is almost independent of κ (for $\kappa < 0{\cdot}4$).

6·1·5. *Solution when* $z_1 = -\kappa$

When $\phi_p > 90°$ the initial value of z may happen to be equal to

$$-\kappa = -F\cos\phi_p/(1-0{\cdot}258\cos^2\phi_p).$$

In this special case dz/da is zero, from (95), and z remains constant at $-\kappa$ as the orbit contracts. The situation is analogous to a circular orbit in a spherical atmosphere, and only

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the variations of a and T with time arise. From equation (31), on replacing suffix 0 by suffix 2, we have

$$\Delta a = -2\pi\delta a^2\rho_2 \exp\{\beta(r_2-a)\}\,[I_0(-\kappa)+I_1(-\kappa)\,F\cos\phi_p+O(e,\kappa e)]. \tag{118}$$

This is the same as the equation for a circular orbit in a spherical atmosphere (King-Hele 1964), if ρ_c, the density at the initial height for a circular orbit, is replaced by

$$\rho_2\{I_0(-\kappa)+\kappa I_1(-\kappa)\},$$

and a_c, the initial radius, is replaced by r_2. From (1), ρ_2 is the density at $\phi = 90°$ at distance r_2 from the Earth's centre: in practice r_2 would be taken equal to a_2, the initial value of a. Thus the equations already derived for circular orbits in a spherical atmosphere remain valid for an orbit with $z = -\kappa$, if ρ_c and a_c are suitably modified in the equations where they appear.

6·1·6. *An alternative solution for r_p in terms of z* $(1 \leqslant z \leqslant 3)$

Equation (99) is a highly satisfactory solution for r_p in terms of z, since the parameter κ has been absorbed into ζ, the argument of the Bessel function. The very generality of this solution does however obscure the dependence of r_p on κ. So it is also useful to derive a form in which the κ term is separated out. In this section the distinction between the definitions of κ does not arise because $z > 1$ and $(z+\kappa)$ cannot be near zero; thus κ is being used as a symbol for $F\cos\phi_p$.

Our procedure is to expand (90) in powers of κ, on the assumption that κ is less than about 0·3. Such an expansion is most likely to be useful if $\frac{1}{2}(I_0+I_2)$ does not much exceed I_1. Now $(I_0+I_2)/2I_1$ increases from 0·90 at $z=3$ to 1·24 at $z=1$, and then tends to infinity as $z\to 0$. So a power series in κ is promising for $1\leqslant z\leqslant 3$. Expanding (90) in powers of κ, and writing $y_n = I_n/I_1$ $(n \neq 1)$, we have

$$\frac{1}{H}\frac{\mathrm{d}a}{\mathrm{d}z} = y_0+\kappa[1-\tfrac{1}{2}\kappa(y_0+y_2)]\,\{1-\tfrac{1}{2}y_0(y_0+y_2)\}+O(e,\kappa^3). \tag{119}$$

As z decreases from 3 to 1, (y_0+y_2) increases from 1·8 to 2·48, so that we may write $y_0+y_2 = 2\{1+O(\kappa)\}$ within the term in square brackets in (119), to give

$$\frac{1}{H}\frac{\mathrm{d}a}{\mathrm{d}z} = y_0+\kappa(1-\kappa)\,\{1-\tfrac{1}{2}y_0(y_0+y_2)\}+O(e,\kappa^3). \tag{120}$$

On replacing the y_n by I_n/I_1, equation (120) may be rewritten as

$$\frac{1}{H}\frac{\mathrm{d}a}{\mathrm{d}z} = \frac{I_0}{I_1}+\kappa(1-\kappa)\left\{1-\frac{I_0}{I_1^2}\frac{\mathrm{d}I_1}{\mathrm{d}z}\right\}+O(e,\kappa^3), \tag{121}$$

where (y_0+y_2) has been eliminated by means of the recurrence relation (Watson 1958, p. 79)

$$I_0+I_2 = 2(\mathrm{d}I_1/\mathrm{d}z).$$

Equation (121) may be integrated at once to give

$$\frac{a_1-a}{H} = \left[\ln\frac{z_1 I_1(z_1)}{zI_1}+\kappa(1-\kappa)\left\{y_{01}-\frac{I_0}{I_1}\right\}\right][1+O(e,\kappa^3)], \tag{122}$$

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where $y_{01} = I_0(z_1)/I_1(z_1)$. Subtracting $(z_1 - z)$ from both sides of (122), we have the required equation for perigee distance r_p

$$\frac{r_{p1}-r_p}{H} = \ln\frac{z_1 I_1(z_1)}{zI_1} - (z_1 - z) - \kappa(1-\kappa)\left\{\frac{I_0}{I_1} - y_{01}\right\} + O(e, \kappa^3). \tag{123}$$

From (122) the orbital period T is given by

$$\frac{T}{T_1} = \left(\frac{a}{a_1}\right)^{\frac{3}{2}} = 1 - \frac{3H}{2a_1}\ln\frac{z_1 I_1(z_1)}{zI_1} + \frac{3\kappa(1-\kappa)H}{2a_1}\left(\frac{I_0}{I_1} - y_{01}\right) + O\left(\frac{eH}{a_1}, \frac{H\kappa^3}{a}\right). \tag{124}$$

6·2. *Solution when ϕ_p is variable* $(1 \leqslant z \leqslant 3)$

6·2·1. *Perigee distance and orbital period in terms of z*

When ϕ_p, and hence κ, is variable, a direct integration of equation (95) is not feasible and we use the alternative approach described in § 6·1·6, which is valid for $z \geqslant 1$. Again $\kappa = F\cos\phi_p$ for all ϕ_p. We start from equation (119) and seek an approximation for the term within braces. Now it turns out, very conveniently, that for $1 \leqslant z \leqslant 3$

$$\tfrac{1}{2}y_0(y_0 + y_2) - 1 \simeq 5/3z^2 \tag{125}$$

with a maximum error of $0{\cdot}06y_0$. With this approximation, (119) becomes

$$\frac{1}{H}\frac{da}{dz} = \left\{y_0 - \frac{5\kappa}{3z^2}\right\}\{1 + O(e, \kappa^2, 0{\cdot}06\kappa)\}. \tag{126}$$

We take the same approximation for ϕ_p as in phase 1, namely equation (65), but rewrite it in the form

$$\cos\phi_p = \cos(P - Q_1 z^2/z_1^2), \tag{127}$$

where P again represents the value of ϕ_p when $z = 0$, and $Q_1 = Qz_1^2/z_0^2$ represents the change in ϕ_p between $z = z_1$ and $z = 0$. The linear variation of ϕ_p with time, which is the basis for (127), is most accurate for $i < 40°$; but in phase 2 the change in ϕ_p will usually be less than in phase 1 because the time is usually shorter, and (127) will then often be an adequate approximation for all inclinations. On writing $\kappa = F\cos\phi_p$ and eliminating ϕ_p by means of (127), equation (126) becomes

$$\frac{1}{H}\frac{da}{dz} = \frac{I_0}{I_1} - \frac{5F}{3z^2}\left(\cos P\cos\frac{Q_1 z^2}{z_1^2} + \sin P\sin\frac{Q_1 z^2}{z_1^2}\right) + O(e, \kappa^2, 0{\cdot}06\kappa). \tag{128}$$

The F-term may be integrated by using the Fresnel integrals C and S defined in (68), to give

$$\frac{a_1 - a}{H} = \ln\frac{z_1 I_1(z_1)}{zI_1(z)} + X + O(e, \kappa^2, 0{\cdot}06\kappa), \tag{129}$$

in which

$$X = \frac{5F}{3z_1}\left[\!\left[\cos\phi_{p1} - \frac{z_1}{z}\cos\phi_p + \pi u_1[\{S(u_1) - S(u)\}\cos P - \{C(u_1) - C(u)\}\sin P]\right]\!\right], \tag{130}$$

where, in analogy with (69), $\quad u_1 = (2Q_1/\pi)^{\frac{1}{2}}, \quad u/u_1 = z/z_1.$ (131)

The corresponding equation for perigee distance is

$$\frac{r_{p1}-r_p}{H} = \left(\frac{r_{p1}-r_p}{H}\right)_{\text{sph. atm.}} + X + O(\kappa^2, 0{\cdot}06\kappa), \tag{132}$$

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while the orbital period is given by

$$\frac{T}{T_1} = \left(\frac{T}{T_1}\right)_{\text{sph. atm.}} - \frac{3HX}{2a_1} + O\left(\frac{H\kappa^2}{a}\right). \tag{133}$$

6·2·2. *The parameter z in terms of time*

We next determine the variation of eccentricity with time by finding how z varies with time. From equation (32) we have

$$\frac{dx}{dt} = -\frac{2\pi}{T}\delta a^2\rho_1 \exp\{\beta(r_1-a)\}\left[I_1+\tfrac{1}{2}F\cos\phi_p(I_0+I_2)+O(e,\kappa^2)\right]. \tag{134}$$

Substituting for $\exp(-\beta a)$ from (129), inverting, and using the approximation

$$\frac{1}{2}\left(\frac{I_0}{I_1}+\frac{I_2}{I_1}\right) \simeq \frac{3}{4}+\frac{1}{2z}, \tag{135}$$

which has a maximum error of 0·07, we can rewrite equation (134) as

$$-\frac{a_1^2 B_2}{H^2}\frac{dt}{dz} = z\left[\!\left[1-F\left(\frac{3}{4}-\frac{7}{6z}\right)\cos\left(P-\frac{Q_1 z^2}{z_1^2}\right)-\frac{5F}{3z_1}\cos\phi_{p1}\right.\right.$$
$$\left.\left.-\frac{5\pi u_1 F}{3z_1}\left[\{S(u_1)-S(u)\}\cos P-\{C(u_1)-C(u)\}\sin P\right]+O(e,\kappa^2,0{\cdot}07\kappa)\right]\!\right], \tag{136}$$

where $$B_2 = \frac{2\pi\delta}{T_1}\rho_1 x_1 I_1(z_1)\exp\{\beta(r_1-a_1)\}. \tag{137}$$

On integrating (136) between the limits z and z_1, we find

$$\frac{2B_2\tau}{e_1^2} = \left(1-\frac{z^2}{z_1^2}\right)\left[1-\frac{5F}{3z_1}\cos\phi_{p1}-\frac{5F\pi u_1}{3z_1}S(u_1)\cos P-C(u_1)\sin P\}\right]$$
$$+\frac{3F}{4Q_1}(\sin\phi_{p1}-\sin\phi_p)+\frac{2F}{3z_1u_1}\left[\{C(u_1)-C(u)\}\cos P+\{S(u_1)-S(u)\}\sin P\right]$$
$$+\frac{5F\pi u_1}{3z_1}\left[\left\{S(u_1)-\frac{u^2}{u_1^2}S(u)\right\}\cos P-\left\{C(u_1)-\frac{u^2}{u_1^2}C(u)\right\}\sin P\right]$$
$$+\frac{5F}{3z_1}\left(\cos\phi_{p1}-\frac{z}{z_1}\cos\phi_p\right)+O(e,\kappa^2), \tag{138}$$

where $\tau = t-t_1$, as usual. Let the value of τ when $e = 0$ be denoted by τ_L, so that

$$\frac{2B_2\tau_L}{e_1^2} = 1+\frac{3F}{4Q_1}(\sin\phi_{p1}-\sin P)+\frac{2F}{3z_1u_1}\{C(u_1)\cos P+S(u_1)\sin P\}. \tag{139}$$

(τ_L differs slightly from the true lifetime, since the final value of e is likely to be non-zero, as § 6·1 indicates). Dividing (138) by (139), we obtain

$$\frac{\tau}{\tau_L} = 1-\frac{z^2}{z_1^2}+\frac{z^2}{z_1^2}X+\frac{3F}{4Q_1}\left\{\sin P-\sin\phi_p-\frac{z^2}{z_1^2}(\sin P-\sin\phi_{p1})\right\}$$
$$-\frac{2F}{3z_1u_1}\left[\left\{C(u)-\frac{z^2}{z_1^2}C(u_1)\right\}\cos P+\left\{S(u)-\frac{z^2}{z_1^2}S(u_1)\right\}\sin P\right]+O(e,\kappa^2), \tag{140}$$

where X is given by (130).

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6·2·3. *Lifetime in terms of* $\dot{T}_1$

Finally, we derive an expression for τ_L in terms of $\dot{T}_1$. From (60) and (31) we have

$$\dot{T}_1 = -3\pi\delta a_1\rho_1 \exp\{\beta(r_1-a_1)\}\,[I_0(z_1)+I_1(z_1)F\cos\phi_{p1}+O(e_1)], \tag{141}$$

or, by (137),

$$\dot{T}_1 = -\frac{3B_2T_1}{2e_1}\{y_{01}+F\cos\phi_{p1}+O(e_1)\}. \tag{142}$$

Multiplying (142) by (139), we obtain

$$\tau_L = -\frac{3e_1T_1}{4\dot{T}_1}y_{01}\Big[1+\frac{F}{y_{01}}\cos\phi_{p1}+\frac{3F}{4Q_1}(\sin\phi_{p1}-\sin P) + \frac{2F}{3z_1u_1}\{C(u_1)\cos P+S(u_1)\sin P\}+O(e_1,\kappa^2)\Big]. \tag{143}$$

If ϕ_p is likely to vary by more than 1 revolution, it is natural when estimating lifetime to 'correct' $\dot{T}_1$ to a value appropriate to the mean density (when $\phi_p = 90°$), i.e. to replace $\dot{T}_1$ by $(\dot{T}_1)_{90} = \dot{T}_1/\{1+(F/y_{01})\cos\phi_{p1}\}$ from (141). Thus

$$\tau_L = (\tau_L)_{\text{sph. atm.}}\Big[\Big[1+F\Big[\frac{3}{4Q_1}(\sin\phi_{p1}-\sin P)+\frac{2}{3z_1u_1}\{C(u_1)\cos P+S(u_1)\sin P\}\Big]\Big]\Big], \tag{144}$$

where

$$(\tau_L)_{\text{sph. atm.}} = -\frac{3e_1T_1y_{01}}{4(\dot{T}_1)_{90}}.$$

When Q_1 exceeds 2π, τ_L never differs from $(\tau_L)_{\text{sph. atm.}}$ by more than about 1 %.

6·3. *Solution when* ϕ_p *is variable* $(z<1)$

As z decreases towards zero, ϕ_p not only varies regularly under the influence of orbital precession, but it is also affected by the day-to-night variation in atmospheric density itself: for example, whenever z passes through zero (and this may happen several times), perigee is likely to re-appear at a point nearly opposite the diurnal bulge, as indicated in § 6·1·1. Thus, when z is near zero, ϕ_p may undergo very rapid changes which depend on the particular value that ϕ_p happens to have as z approaches zero. In these circumstances the variation of ϕ_p with z cannot be specified beforehand. A quite different treatment is called for, and it has been decided to discuss the theory for $z<1$ in a separate paper.

Although there is certainly a need to evaluate the variation of z and ϕ_p with t when z is small, it is worth noting that the semi major axis a is a more important parameter than z, because the position of perigee and the value of the eccentricity do not have much influence on the orbital contraction for near-circular orbits. It should therefore be possible to obtain a useful solution for the variation of a with time while taking a mean value for ϕ_p. Specifically, if $z<0·5$ (e less than about 0·003), equation (31) may be rewritten as

$$\Delta a = -2\pi\delta a^2\rho_1 \exp\beta(r_1-a)\,\{1·03+O(0·03,\,0·1F\cos\phi_p,\,e,\,F^2)\}, \tag{145}$$

where the Bessel functions have been replaced by their mean values over the interval $0<z<0·5$. Equation (145) is the same as for a circular orbit in a spherical atmosphere (King-Hele 1964) if ρ_c, the density at the initial height for a circular orbit, is, in the few

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equations where it appears, replaced by $1{\cdot}03\rho_1$, where ρ_1 is the value of ρ at mean initial orbital height. The error term 0·01 which appears in the circular-orbit theory would however be increased to 0·03 or $0{\cdot}1\,F\cos\phi_p$, whichever was higher.

6·4. *Formula for air density in terms of* $\dot{T}$

A formula for the mean air density ρ_0 at a distance r_0 from the Earth's centre in terms of $\dot{T}$ can be obtained from equations (31) and (32)

$$\rho_0 = -\frac{\dot{T}}{3\pi\delta a}\,\frac{\exp\{\beta(a-r_0)\}}{I_0+2eI_1+F\cos\phi_p\{I_1+\tfrac{1}{2}e(I_0+3I_2)\}+O(e^2)}. \tag{146}$$

This is a one-revolution result and applies for either fixed or variable ϕ_p. It is worth remembering that there is no restriction on the magnitude of F, provided that the density variation resembles the form given in equation (1).

If r_0 is taken as the perigee distance r_p, as will often be appropriate, equation (146) gives the mean density at perigee height. The exponent $\beta(a-r_0)$ in (146) then reduces to z.

7. Discussion

The aim of this paper has been to extend the theory developed in part I for a spherically symmetrical atmosphere to an atmosphere which exhibits a day-to-night variation in density resembling that in the real terrestrial atmosphere. We assume that the air density ρ at a given distance r from the Earth's centre depends only on the angular distance ϕ from the centre of the diurnal bulge, i.e. the point where density is greatest. We take ρ to vary sinusoidally with ϕ and exponentially with r,

$$\rho = \rho_0(1+F\cos\phi)\exp\{-(r-r_0)/H\},$$

where ρ_0, r_0, F and H are constant. This form, although very simple, is a satisfactory first approximation (figure 2). F may be expected to take values from about 0·1 at 200 km height to about 0·7 at 600 km height (see figure 3). H will usually take a value between 20 and 70 km.

The analysis is in two parts, phase 1 when the orbital eccentricity e lies between about 0·02 and 0·2, and phase 2 when e is less than about 0·02, or, strictly, $z < 3$, where $z = ae/H$. In both phase 1 and phase 2 the results take different forms, according as the angular distance ϕ_p of the perigee point from the centre of the bulge is constant or varies linearly with time.

The variation of perigee distance r_p with eccentricity e in phase 1 for a spherically symmetrical atmosphere is shown in figure 4 of part I. Under the 'constant-ϕ_p' theory (which in practice would probably be used if ϕ_p did not vary by more than about 30°), the correction to be subtracted from $(r_{p0}-r_p)/H$ to allow for the effects of the day-to-night variation in density is $\mu\xi/z_0$, where ξ is shown in figure 5 and $\mu = F\cos\phi_p/(1+F\cos\phi_p)$. When ϕ_p varies with time, the correction Ψ to be added to $(r_{p0}-r_p)/H$, given by equation (72), has as its dominant term

$$\tfrac{1}{2}F\left(\frac{\cos\phi_{p0}}{z_0}-\frac{\cos\phi_p}{z}\right).$$

It is to be expected that the correction to $(r_{p0}-r_p)/H$ will be fairly small when the eccentricity is 0·1 or greater. For most of the drag then occurs near perigee, and the details of the

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variation in drag at points distant from perigee are not very important. For example, if $\phi_p = 0$, it is of little consequence that the drag at apogee is $\frac{1}{5000}$ of that at perigee, instead of, say, $\frac{1}{1000}$ in a spherically symmetrical atmosphere. If the initial eccentricity is smaller, however, the effect of the day-to-night variation can be much greater.

Alternatively, the effect of the day-to-night variation in density can be allowed for (with constant ϕ_p) by continuing to use the spherical-atmosphere results, but with $(e+\mu H/a_0)$ instead of e. If, for example, $e_0 = 0{\cdot}1$, $(r_{p0}-r_p)/H = 0{\cdot}5$ and $H/a = 0{\cdot}006$, then $e = 0{\cdot}037-0{\cdot}0038\mu$, which shows directly how e varies with μ for a given value of $(r_{p0}-r_p)/H$.

The variation of eccentricity with time when ϕ_p is constant is given by equation (58) and shown in figure 6. This graph applies both for phase 1, reading numbers on the curves as values of μ/z_0, and also for phase 2, if the numbers on the curves are read as values of $\kappa/z_1(=F\cos\phi_p/z_1)$. The day-to-night variation in density can have quite a large effect. Half-way through the life, for example, e/e_0 is 0·71 for a spherically symmetrical atmosphere, but would be reduced to 0·67 if $\mu/z_0 = 0{\cdot}1$, e.g. if $z_0 = 5$ and $\mu = 0{\cdot}5$.

Alternatively the day-to-night variation in air density can be allowed for by continuing to use the spherical-atmosphere result, namely $e^2/e_0^2 = 1-t/t_L$, but with $(e+\mu H/a_0)$ instead of e, so that in phase 1

$$\left(\frac{e+\mu H/a_0}{e_0+\mu H/a_0}\right)^2 = 1-\frac{t}{t_L}.$$

In phase 2, a similar equation applies, e being replaced by $(e+\kappa H/a_1)$, but certain complications (to be discussed later) arise when $e = 0$.

For phase 1 and ϕ_p constant, the day-to-night variation in density can considerably alter the formula for lifetime in terms of $\dot{T}$, as figure 7 shows. If $\mu = -1$, to take an extreme example, the lifetime is shorter by between 3 % (for $e_0 = 0{\cdot}2$) and 30 % (for $e_0 = 3H/a_0$) than would be predicted on the basis of a spherically symmetrical atmosphere. This effect is to be expected: for, if the perigee is 'stuck' at $\phi_p = 180°$, the apogee height will decrease at a fairly normal rate until the orbit is near-circular; by then the density at apogee will be much greater than would have been expected for a spherical atmosphere and the subsequent decay will be much quicker. If ϕ_p varies by more than 1 cycle, however, the day-to-night variation in density has little effect on the lifetime formula, provided the value of $\dot{T}$ used is corrected to the value appropriate to $\phi_p = 90°$, or averaged over a cycle of ϕ_p.

If phase 2, as is usual, lasts for a shorter time than phase 1, ϕ_p is less likely to show a large variation and the constant-ϕ_p solution is of more importance than in phase 1. The constant-ϕ_p solution is particularly simple: it is found that the spherical-atmosphere results still hold if z is replaced by $z+\kappa$, where $\kappa \simeq F\cos\phi_p$. (For $\phi_p < 90°$, $\kappa = F\cos\phi_p$; and for $\phi_p > 90°$, $\kappa = F\cos\phi_p/(1-0{\cdot}258F^2\cos^2\phi_p)$.) κ is assumed to be less than 0·4. Thus the graph for the variation of r_p, figure 10, is the same as before, except that the variable has been changed from z to $\zeta = z+\kappa$. With κ having values up to 0·4, figure 10 shows that the day-to-night variation in density can have a large effect on the value of r_p for given eccentricity, that is, for given z: for $z = 0{\cdot}6$ and $z_1 = 3$, $(r_{p1}-r_p)/H$ changes from 1·7 when $\kappa = 0$ to 3·4 when $\kappa = -0{\cdot}4$.

The lifetime of the satellite ends when z approaches the value which makes $(r_{p0}-r_p)$ tend to infinity. In parts I–IV this always occurred as $z\to 0$, but now it occurs as $\zeta\to 0$

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(see figure 10). Thus the lifetime ends just before z reaches the value $-\kappa$, instead of just before z reaches zero. There are two possible situations:

(1) If $\phi_p > 90°$, κ is negative, so that the lifetime ends as the eccentricity tends towards a finite positive value, when z is approximately equal to $-\kappa \simeq -F\cos\phi_p$, and $e \simeq -HF\cos\phi_p/a_1$. If $\phi_p = 180°$, this means that at the end of the life the orbit tends towards an eccentricity equal to that of the atmosphere, so that drag is approximately constant round the orbit.

(2) If, on the other hand, $\phi_p < 90°$ (so that $\kappa > 0$), the eccentricity can decrease to zero without causing an excessive decrease in perigee height, since $\zeta = \kappa$ when $e = 0$. Thus, for an orbit with $\kappa > 0$, the eccentricity decreases until the orbit is circular, and then a new perigee develops near the point of minimum density on the circular orbit. ϕ_p therefore undergoes a discontinuity at $z = 0$ and in the 'post-circular' phase takes a new value between 90° and 180° (this value is always 180° if the orbit passes through the $\phi = 180°$ point at the time it is circular). Thus κ takes a new (negative) value, and ζ increases from a negative value towards zero, as on the right-hand branch of figure 10.

Figure 11 shows situations (1) and (2) in practice, for $F = 0{\cdot}2$: for $\phi_p = 180°$, z decreases steadily towards a value close to 0·2; for $\phi_p = 0$, z decreases to zero, and, since the orbit passes through $\phi = 180°$ when $\phi_p = 0$, the post-circular value of ϕ_p is 180°, and the post-circular value of κ is $-0{\cdot}2$, so that again $z \to 0{\cdot}2$ at the end of the life.

An interesting special case occurs when $z = -\kappa$ ($\zeta = 0$) initially. The distribution of drag round the orbit is then such that the eccentricity has no tendency to change: the value of z therefore remains constant at $-\kappa$ as the orbit contracts, and only the variation of semi major axis with time arises. This is analogous to a circular orbit in a spherically symmetrical atmosphere, and the results are the same.

For a spherically symmetrical atmosphere the square of the eccentricity (or of z) varies almost linearly with time. When the day-to-night variation in density is allowed for, with ϕ_p constant, it is ζ^2 which varies linearly with time, as equation (108) shows. If z passes through zero, however, equation (108) must be used in two stages. Suppose, for example, that $z_1 = 0{\cdot}3$, $F = 0{\cdot}2$, $\phi_p = 0$ and the lifetime is 10 days: then we have $\zeta = z + 0{\cdot}2$ and from equation (110) the time at $z = 0$, τ_z, is given by $\tau_z = 10\{1-(0{\cdot}2/0{\cdot}5)^2\} = 8{\cdot}4$ days. If $\tau < \tau_z$, z can be found from equation (108) with $\zeta = z + 0{\cdot}2$. If $\tau > \tau_z$, time must be measured from τ_z as zero point and, since $\kappa = -0{\cdot}2$ in the post-circular phase, $\zeta = z - 0{\cdot}2$. Thus, for example, at time 9·2 days, $(\tau-\tau_z)/(\tau_L-\tau_z) = 0{\cdot}50$ and $\zeta/\zeta_1 = 0{\cdot}71$, giving $z = 0{\cdot}058$ ($e = 0{\cdot}058H/a$).

Figure 12 gives the lifetime in terms of $\dot{T}_1$ for phase 2, constant ϕ_p. This is the same as for a spherical atmosphere, except that z_1 is replaced by ζ_1. If $|\kappa| < 0{\cdot}4$, the maximum change in lifetime resulting from the day-to-night variation is about 10 %. For an initially circular orbit the day-to-night variation in density has no effect on lifetime, apart from terms of order $\kappa^2/8$.

Full results for variable ϕ_p in phase 2 are given only for $z > 1$. The change in perigee height due to the day-to-night variation can again be substantial: if $F = 0{\cdot}2$ and $z_1 = 2$, the correction to be added to $(r_{p1}-r_p)/H$ is of order 0·1 when $z = 1$, while $(r_{p1}-r_p)/H$ itself is of order 0·7.

The above discussion indicates that the effect of a day-to-night variation depends on

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a large number of parameters and this makes it difficult to draw general conclusions. As mentioned previously, many of the equations have the same form as those for a spherical atmosphere derived in part I, if the eccentricity is replaced by $(e+\mu H/a_0)$ in phase 1 and z is replaced by $(z+\kappa)$ in phase 2. Thus, for orbits with $\mu H/a_0$ small compared with e, that is with F/z small, it follows that, whether ϕ_p varies or not, the day-to-night variation has little effect and e^2 still varies almost linearly with time. This result also suggests that F/z provides perhaps the best general indication of the severity of the day-to-night effect on the orbit.

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